Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 2: Structural Properties of Preferences and Utility Functions

Summary of the Chapter

This chapter concerns structural properties that preferences and their utility functions might have. The setting, for the sake of definiteness, is $X = R^k_+$, but many of the results extend to more general spaces. The chapter provides three types of results:

1(a). *If preferences $\succeq$ have property A, then **every** utility function u that represents $\succeq$ has property Z.*

1(b). *If preferences $\succeq$ have property A, then **some** utility function u that represents $\succeq$ has property Z.*

2. *If preferences $\succeq$ are represented by a utility function u that has property Z, then $\succeq$ has property A.*

The categories for property A include three of the most important sets of structural assumptions in economic theory, namely *monotonicity*, *convexity*, and *continuity*. It also includes properties that are useful in the analysis of specific economic problems; the buzzwords here are *separability*, *quasi-linearity*, and *homotheticity*.

A summary of the results of this form is given in Table G2.1. (This is the solution to Problem 2.1 from the text; you might want to try that problem on your own before consulting this figure.) For example, the first line reads *Preferences are monotone implies that **all** utility function representations (of those preferences) are nondecreasing* and *Preferences are monotone is implied if **some** utility function representation is nondecreasing.* I give

Preferences are	implies	is implied if	utility function representation(s) is/are	
monotone	all	some	nondecreasing	Def. 2.1; Prop. 2.2
strictly monotone	all	some	strictly increasing	
convex	no implication (see note 1)	some	concave	Defs. 2.4 & 2.7; Prop. 2.8(a)
strictly convex		some	strictly concave	
convex	all	some	quasi-concave	Defs. 2.4 & 2.7; Prop. 2.8(b)
semi-strictly convex	all	some	semi-strictly quasi-concave	
strictly convex	all	some	strictly quasi-concave	
continuous	some	some	continuous	Def. 1.13; Prop. 2.9 (Debreu's Theorem)
weakly separable	all	some	weakly separable form (see note 2)	Def. 2.11; Prop. 2.12
strongly separable	some (see n. 6)	some	additively separable (see note 3)	Def. 2.13; Prop. 2.14
such that they satisfy the 3 properties in Prop. 2.16	some	some	quasi-linear form (see note 4)	Def. 2.15; Prop. 2.16
continuous and homothetic (see note 5)	some	some	continuous and homogeneous	Defs. 2.17 & 2.18; Prop. 2.19

Table G2.1. The heart of Chapter 2. (See notes below.)

Notes: 1. Do not attempt to fill in the box marked ????. You were not given this information in the chapter. (In fact, I don't know what is correct to put in this box.)

2. u takes the form $v(u_1(x_{J_1}), u_2(x_{J_2}), \ldots, u_N(x_{J_N}), x_{K\setminus(J_1\cup\ldots\cup J_N)})$ for v strictly increasing in its first N arguments.

3. u takes the form $\sum_{n=1}^{N} u_n(x_{J_n})$.

4. U takes the form $u(x) + m$.

5. In this row, a maintained hypothesis is that preferences are continuous.

6. Proposition 2.14, which is the basis for this "some," also requires that preferences are continuous and that there are at least three nontrivial dimensions.

the results in the table and, in the right-hand margin, I refer you to the appropriate definitions and propositions from the text.

Note that in the direction of implication from the utility function to the representation—results of type 2—it is always the case that if *some* representation has the property, preferences have the corresponding property. But in the other direction, sometimes the implication is that *all* representations have the corresponding property, and sometimes it is that *some* representation does so. (Note that for convexity of preferences and concavity of the utility function, we don't even have *some*.) In this regard, you should convince yourself that if there is an *all* in the second column, then the property in the fourth column is preserved under strictly increasing rescalings (and vice versa). That is, if $u : X \to R$ is a nondecreasing function, and if $f : R \to R$ is strictly increasing, then v defined by $v(x) = f(u(x))$ is also nondecreasing, because in the second column of row 1 we have an *all*. While in the second column of row 8, we have *some*, so we know that continuity of u is not necessarily preserved if we take a strictly increasing rescaling.

These are useful results primarily because the properties in the fourth column, properties of a representing utility function, are convenient for proving results. Take Proposition 1.19, for example, which states that if preferences are continuous, choice from a nonempty compact set A is nonempty. This can be proved from first principles based on the definition of continuous preferences, but it is easier to go by way of a continuous representation and the standard result from mathematics that a continuous function achieves its maximum on a nonempty compact set. The point is, if we want to assume that a utility representation of preferences has a property such as continuity, quasi-concavity, or quasi-linearity, we must answer the question, What does this entail about the underlying preferences? The results summarized in Figure G2.1 give us answers.

Why do we care? Continuity and convexity are important because they provide us with very useful characterizations of the set of solutions of (constrained) choice problems:

- If preferences $\succeq$ are continuous and A is a nonempty and compact set, then $c_{\succeq}(A)$ is nonempty (Proposition 1.19).
- If preferences $\succeq$ are convex and A is a convex set, then $c_{\succeq}(A)$ is a convex set (Proposition 2.6).
- If preferences $\succeq$ are strictly convex and A is a convex set, then $c_{\succeq}(A)$ contains either one element or is empty (also Proposition 2.6).

Monotonicity of preferences and the related properties of insatiability will play an important role in the theory of the consumer; anticipating developments in the next chapter, if preferences are locally insatiable, a consumer choosing a consumption bundle subject to a budget constraint will spend all her income. Separability, quasi-linearity, and homotheticity all play important simplifying roles in specific problems; this happens beginning next chapter and then throughout the book.

Solutions to Starred Problems

■ 2.1. (a) See Table G2.1.

(b) Lexicographic preferences (which I'll write $\succeq_L$) are strictly monotone, strictly convex, and weakly separable. None of these are hard to prove, so I won't spell out the details, but it is interesting to note, at least, that lexicographic preferences are *anti-symmetric*, meaning that if $x \succeq_L y \succeq_L x$, then $x = y$. So if $x \succeq_L y$ and $x \neq y$, then $x \succ_L y$. This in turn implies (if $x = (x_1, x_2)$ and $y = (y_1, y_2)$) that either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. In either case, for all $\alpha \in (0, 1)$, $x \succ_L \alpha x + (1 - \alpha)y \succ_L y$, which of course gives us strict convexity. Each of these three properties of preference relations imply *all* representing utility functions have the properties given in the right-hand column (and corresponding row) of Table G2.1, which is consistent with the fact that lexicographic preferences have *no* numerical (utility) representation.

We have *some* in the second-from-left-hand columns for continuity, strong separability, the 3 properties of Proposition 2.16, and continuous and homothetic preferences. So, since lexicographic preferences have no numerical representation, they must not have any of these properties. It is easy to see that lexicographic preferences are not continuous, which takes care of the first and fourth of these four. Concerning the three properties in Proposition 2.16, lexicographic preferences satisfy properties a and b. But they fail to satisfy c: If $x > x'$, then no amounts of the second component can compensate for this difference in the first component.

This leaves strong separability. As note 6 in Table G2.1 states, the *some* for that row requires that preferences are strongly separable *and continuous*, and that there are at least three nontrivial components. Lexicographic preferences as defined in Problem 1.10 were for a subset of R^2, so they fail on the three-nontrivial-component requirement. But it is easy to define lexicographic preferences for, say, R^n_+, where $n \geq 3$, and then there will be three or more nontrivial components. And, it is not hard to show, they satisfy Definition 2.13; that is, they are strongly separable (into individual components). So the "problem," so to speak—the reason that there is *some* in the second-from-left column, which implies the existence of a representation, while lexicographic preferences have no representation—is that lexicographic preferences are not continuous.

■ 2.3. Suppose that $\succeq$ is globally insatiable and semi-strictly convex. Let x be any point from X. Then by global insatiability, there is some $x' \in X$ such that $x' \succ x$. For any $\epsilon > 0$, there is some $a \in (0,1)$ (very close to zero) such that the distance from x to $ax' + (1-a)x$ is less than ϵ. But by semi-strict convexity, for every $a \in (0,1)$, $ax' + (1-a)x \succ x$. Thus preferences are locally insatiable.

Suppose $k = 1$. Define $\succeq$ on $X = R_+$ as the preferences represented by the following utility function:

$$u(x) = \begin{cases} x, & \text{if } x \leq 1, \\ 1, & \text{if } 1 \leq x \leq 2, \text{ and} \\ x-1, & \text{if } x \geq 2. \end{cases}$$

Because these preferences give convex NWT sets, they are convex. And because u is unbounded above, preferences are globally insatiable. But preferences are not locally insatiable at, for example, $x = 3/2$. Being very pedantic, the problem is that convex preferences (without semi-strict convexity) can be "flat" for a while, which is incompatible with local insatiability.

■ 2.5. We give the same example for parts a and b. Let $k = 1$, and let $u(x) = x$. This is certainly concave and continuous. Now let f be given by

$$f(r) = \begin{cases} (r-1)^3, & \text{for } 0 \le r \le 2, \text{ and} \\ r, & \text{for } r > 2. \end{cases}$$

By inspection, f is strictly increasing. Also, $f(u(x)) = f(x)$. But f is neither concave nor continuous; it is convex over the region $x \in [1,2]$, and it is discontinuous at $x = 2$. (It also has derivative zero at $x = 1$, which is something you may want to remember for problems upcoming in later chapters.)

■ 2.6. Suppose preferences are monotone and locally insatiable. Take any two points x and x' such that x' is strictly greater than x, component by component. Let $\epsilon = \min_{i=1,\ldots,k} x'_i - x_i$; then $\epsilon > 0$. Moreover, by construction, if x'' is within ϵ of x, then $x'' \le x'$. (If $x''_i > x'_i$ for some i, then $x''_i - x_i > \epsilon$, which implies that $\|x'' - x'\| > \epsilon$.) Use local insatiability to produce some x'' within ϵ of x such that $x'' \succ x$. Since $x'' \le x'$, by monotonicity, $x' \succeq x''$, and thus $x' \succ x$. This shows that preferences are strictly monotone for strict increases.

Conversely, suppose that preferences are strictly monotone for strict increases. Local insatiability is obvious: For any x and $\epsilon > 0$, let $x' = x + (\epsilon/k, \epsilon/k, \ldots, \epsilon/k)$. Then x' is strictly greater than x, then $x' \succ x$. By construction, $\|x' - x\| = \epsilon/\sqrt{k} \le \epsilon$. Monotonicity requires continuity. Suppose $x' \ge x$. Let $x^n = x' + (\frac{1}{n}, \ldots, \frac{1}{n})$; then x^n is strictly greater than x, hence $x^n \succ x$. By continuity, $\lim_{n\to\infty} x^n \succeq x$, but this limit is x', hence preferences are monotone.

Without continuity, preferences that are monotone and locally insatiable are strictly monotone for strict increases—the proof in the first paragraph works—and preferences that are strictly monotone for strict increases are locally insatiable, by virtue of the first part of the second paragraph. But preferences that are strictly monotone for strict increases need not be monotone. To see this, let $k = 2$ and define

$$u((x_1, x_2)) = \begin{cases} 0, & \text{if } x_1 = x_2 = 0, \\ -1, & \text{if } \max\{x_1, x_2\} > \min\{x_1, x_2\} = 0, \text{ and} \\ \min\{x_1, x_2\}, & \text{if } \min\{(x_1, x_2\} > 0. \end{cases}$$

In words, this utility function gives the "standard" preferences with right-angle indifference curves (along the 45° line) in R^2_+, except that all points along the two axes, except 0, are worse than 0. If you compare any two points x and x', one of which is strictly greater than the other, then the strictly greater point is off the axes, and it is simple to see that its utility exceeds that of the first. So these preferences are strictly monotone for strict increases. But clearly preferences are not monotone, since $(0, 0) \succ (0, 1)$.

■ 2.9. (a) For preferences given by the utility function $u((x_1, x_2)) = \alpha x_1 + \beta x_2$ $(\alpha, \beta > 0)$, the indifference curves are the loci of points $\alpha x_1 + \beta x_2 = c$, or $x_2 = (c - \alpha x_1)/\beta$, for all

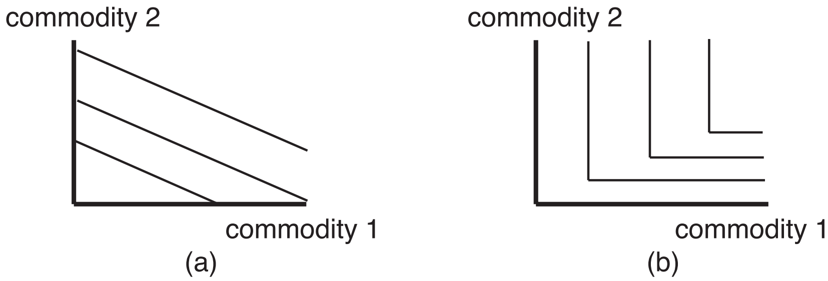

Figure G2.2. Problem 2.9: Two sets of indifference curves

constants c. These are parallel straight lines, with slope $-\alpha/\beta$; the picture is in Figure G2.2(a).

These preferences are: strictly monotone (since u is strictly increasing) and therefore increasing; locally insatiable (since they are strictly monotone); convex (since the NWT sets are convex); not strictly convex (indifference curves are linear); semi-strictly convex (if $x \succ y$, then $u(x) > u(y)$, and since u is linear, $u(ax + (1-a)y) = au(x) + (1-a)u(y) > u(y)$ for $a > 0$); and continuous (since u is continuous).

(b) For preferences given by the utility function $u((x_1, x_2)) = \min\{x_1/\alpha, x_2/\beta\}$, indifference curves are loci of points satisfying $\min\{x_1/\alpha, x_2/\beta\} = c$, or $[x_1 = \alpha c$ and $x_2 \geq \beta c]$ or $[x_1 \geq \alpha c$ and $x_2 = \beta c]$, for all constants c. These indifference curves are "right angles" whose angle comes at points along the line $x_1/\alpha = x_2/\beta$ or $x_2 = \beta x_1/\alpha$. The picture is in Figure G2.2(b).

These preferences are monotone (u is increasing) but not strictly monotone (for example, $(\alpha, \beta) \sim (\alpha + 1, \beta)$); locally insatiable (for any $x = (x_1, x_2)$, $y = (x_1 + \epsilon/2, x_2 + \epsilon/2)$ is within ϵ of x and is strictly better than x); convex (since the NWT sets are convex); not strictly convex (look at convex combinations of (α, β) and $(\alpha + 1, \beta)$); semi-strictly convex (proof given momentarily); and continuous (u is continuous).

The one slightly difficult thing to prove is that these preferences are semi-strictly convex. Suppose $x = (x_1, x_2) \succ y = (y_1, y_2)$. Then $x_1/\alpha \geq u(x) > u(y)$, $x_2/\beta \geq u(x) > u(y)$, $y_1/\alpha \geq u(y)$, and $y_2/\beta \geq u(y)$. Therefore, for any $a \in (0, 1)$,

$$\frac{ax_1 + (1-a)y_1}{\alpha} > u(y) \quad \text{and} \quad \frac{ax_2 + (1-a)y_2}{\beta} > u(y).$$

This implies that $u(ax + (1-a)y) > u(y)$, or $ax + (1-a)y \succ y$.

■ 2.11. Proving that a quasi-linear representation implies properties a, b, and c is entirely straightforward, so I'll only give the proof that the three properties imply the representation.

Begin by observing that property a implies: *If $m > m'$, then $(x, m) \succ (x, m')$ for all x. And if $(x, m) \succ (x, m')$ for some x, then $m > m'$.* For the first part of this, suppose $m > m'$. The the property immediately implies that $(x, m) \succeq (x, m')$. So we only need to rule out $(x,m) \sim (x,m')$, but by the property, that would imply $m' \geq m$, which is false. Then for the second part, $(x,m) \succ (x,m')$ implies $(x,m) \succeq (x,m')$ implies $m \geq m'$, and $m = m'$ is precluded because if $m = m'$, then $(x,m) = (x,m')$ and hence $(x,m) \sim (x,m')$, which by hypothesis is not true.

Next, observe that property b implies: *For every* $x, x' \in R_+^{K-1}$ *and* m, m', *and* $m'' \in R_+^k$ *such that* $m - m'' \geq 0$ *and* $m' - m'' \geq 0, (x,m) \succeq (x',m')$ *if and only if* $(x, m - m'') \succeq (x', m' - m'')$. This simply involves letting $n = m - m''$ and $n' = m' - m''$ and invoking property b for n and n' in place of m and m'' with $m'', x,$ and x'.

Next observe the following: *If* $(x,m) \sim (x',m')$ *and* $(x,n) \sim (x',n')$, *then* $m - m' = n - n'$. To see this, assume the antecedent. Assume $n \geq m$. Let $\delta = n - m$; by property b from Proposition , $(x,m) \sim (x',m')$ implies $(x, m+\delta) \sim (x', m'+\delta)$. But $m + \delta = n$, and $(x,n) \sim (x',n')$, so that $(x',n') \sim (x', m'+\delta)$. By property a, $m' + \delta = n'$, or $n' - m' = \delta = n - m$. The argument if $m \geq n$ is entirely symmetrical.

Following the hint, fix some x^0. For each x, use c to find m_x and m'_x such that $(x, m_x) \sim (x^0, m'_x)$, and define $u(x) = m'_x - m_x$. The result in the previous paragraph implies that this definition is independent of the particular m_x and m'_x selected; that is, if $(x,n) \sim (x^0, n')$, then $u(x) := m'_x - m_x = n' - n$.

Take any x and $m \geq -u(x)$. Let $\delta = m - m_x$. If $\delta \geq 0$, use property b and $(x, m_x) \sim (x, m'_x)$ to conclude that $(x^0, m'_x + \delta) \sim (x, m_x + \delta)$. But $m_x + \delta = m$, while $m'_x + \delta = m'_x + m - m_x = m + u(x)$. Therefore, $(x^0, m + u(x)) \sim (x, m)$. And if $\delta \leq 0$, use the "extension" of property b given in the second paragraph of this solution to conclude that since $(x, m_x) \sim (x^0, m'_x)$, $(x,m) = (x, m_x + \delta) \sim (x^0, m'_x + \delta) = (x^0, m + u(x))$.

Let $U(x,m) = u(x) + m$. We are done if we show that U gives a representation of $\succeq$. To see this, fix any pair (x,m) and (x',m'). Let $K = \max\{-u(x), -u(x'), 0\}$. By property b, $(x,m) \succeq (x',m')$ if and only if $(x, m+K) \succeq (x, m'+K)$. But $(x, m+K) \sim (x^0, m + K + u(x))$ and $(x', m'+K) \sim (x^0, m' + K + u(x'))$, therefore $(x,m) \succeq (x',m')$ if and only if $(x^0, m + K - u(x)) \succeq (x^0, m' + K + u(x'))$ which, by property a and the first paragraph, is true if and only if $m + K + u(x) \geq m' + K + u(x')$ if and only if $m + u(x) \geq m' + u(x')$.

■ 2.14. We do not provide all the details of the proof, but following are some of the harder steps.

(a) Following the hint given in the text, let V be any numerical representation of $\succeq$. Let $\bar{v} = \sup\{V(x); x \in X\}$, and let $\underline{v} = \inf\{V(x); x \in X\}$.

If there is some $x \in X$ such that $V(x) = \bar{v}$, set z_1 to be any such x. Otherwise, let $\{z_1, z_2, \ldots\}$ be a sequence out of X such that $V(z_n)$ is strictly increasing in n and $\lim_n V(z_n) = \bar{v}$.

If there is some $x \in X$ such that $V(x) = \underline{v}$, set y_1 to be any such x. Otherwise, let $\{y_1, y_2, \ldots\}$ be a sequence out of X such that $V(y_n)$ is strictly decreasing in n and $\lim_n V(y_n) = \underline{v}$.

Let Z be the union of the following pieces: (1) In all cases, the line segment joining y_1 to z_1. (2) If $V(z_1) < \bar{v}$, the line segments joining z_1 to z_2, z_2 to z_3, z_3 to z_4, and so on. (3) If $V(y_1) > \underline{v}$, the line segments joining y_1 to y_2, y_2 to y_3, and so on.

Z is the union of line segments with joined endpoints, so it traces out a continuous one-dimensional "curve" in Z. And by construction, for all $x \in X$, there exist z_n and y_m with $V(z_n) \geq V(x) \geq V(y_m)$ for some n and m, thus $z_n \succeq x \succeq y_m$.

(b) The next step is to apply the lemma to conclude that for every $x \in X$ there is some $z(x) \in Z$ such that $x \sim z(x)$. There is nothing to prove here. The lemma applies directly.

(c) Next we are to prove that if U_Z is a continuous representation of $\succeq$ restricted to Z, we can extend U_Z to all of X to get a continuous representation of $\succeq$ on X.

To show this, note first that by part b, for each x there is some $z(x) \in Z$ such that $x \sim z(x)$. Define $U(x) = U_Z(z(x))$. We must show that the definition doesn't depend on which $z(x) \in Z$ such that $z(x) \sim x$ we pick. If $z \sim x \sim z'$,then $z \sim z'$ and $U_Z(z) = U_Z(z')$. Therefore, U is well defined on X.

Next, to show that U represents $\succeq$, we have: $x \succeq x'$ if and only if $z(x) \succeq z(x')$ (since $x \sim z(x)$ and $x' \sim z(x')$) if and only if $U_Z(z(x)) \geq U_Z(z(x'))$ (since U_Z represents $\succeq$ on Z) if and only if $U(x) \geq U(x')$ (by the definition of U).

To finish part b, we have to show that U is continuous. Suppose that $\{x_n\}$ has limit x in X, but that $U(x_n)$ does not have limit $U(x)$. By looking along a subsequence if necessary, we can assume that, for some $\delta > 0$, either $U(x_n) > U(x) + \delta$ for all sufficiently large n or $U(x_n) < U(x) - \delta$. Consider the former case. Since $U(x_n) = U_Z(z(x_n))$ and $U(x) = U_Z(z(x))$, by the continuity of U_Z and the intermediate-value theorem there is some $z' \in Z$ with $U_Z(z') = U(x) + \delta/2$. Of course, $z' \succ z(x)$ (since $U(z') > U(z(x))$). For all n sufficiently large, $U(x_n) \geq U(z')$, therefore $x_n \succeq z'$, and thus by continuity $x = \lim x_n \succeq z'$. But $x \succeq z' \succ z(x)$ contradicts $x \sim z(x)$. The case where $U(x_n) < U(x) - \delta$ for all sufficiently large n gives a similar contradiction, and U must be continuous.

So it remains is to produce a continuous representation on Z.

Recall what you were told to do next: Let Z' be a countable dense subset of Z. To be specific, let Z' be all the points on the various line segments that make up Z', that have the form $qx + (1 - q)x'$, where x and x' are endpoints of one of the line segments and q is a rational number. Note that this ensures that z^1 and y^1 are both in Z'. For each line segment, there are countably many such convex combinations of the endpoints, and there are countably many line segments; since the countable union of countably many sets is countable, Z' so defined is countable.

(d) This part asks you to show that this procedure produces a numerical representation of $\succeq$ on Z'. Details of this step are left to you, but the idea is: At each step, we produce a representation of $\succeq$ restricted to finite subsets of Z'. To show this use induction on the size of the finite subset. Then the extension to all of Z' gives a numerical representation because for any pair z and z' from Z', values of $U_{Z'}$ for z and z' were defined at some finite step in the procedure.

(e) Next: Let I be the smallest interval containing $U_{Z'}(z')$ for all $z' \in Z'$. We must show that the set of values $\{r = U_{Z'}(z')$ for some $z' \in Z'\}$ is dense in I. This is the trickiest part of the proof and is essential to what follows. It is shown as follows: Suppose it is not true. Then there is an interval J of length greater than zero inside I such that $U_{Z'}(z') \notin J$ for any z'. Let J^* be the largest open interval containing J for which this is so, and let δ be the width of J^*. There are then points z and z' from Z' such that $U_{Z'}(z)$ is above J^* but within $\delta/5$ of J^*'s right-hand endpoint, and $U_{Z'}(z')$ is below J^* but within $\delta/5$ of the left-hand endpoint of J^*. Let m be the larger of the indices of z and z' in the ennumeration of Z'. Consider $\{z'_1, \ldots, z'_m\}$, and let y' be the element of this set with largest $U_{Z'}$ value that is still to the left of J^*, and y be the element of this set with smallest $U_{Z'}$ value but still to the right of J^*. Of course, y and y' are within $\delta/5$ of the interval's nearer endpoint. Also, $y \succ y'$

By continuity of $\succeq$, there will be some subsequent z'_n such that $y \succ z'_n \succ y'$. Assume that n is the first index after m such that this happens. By construction, $U_{Z'}(z'_n) = (U_{Z'}(y) + U_{Z'}(y'))/2$. But again by construction, this average lies in the interval J^* (somewhere between two-fifths and three-fifths along the interval), which contradicts the supposition that J^* is not hit by $U_{Z'}$. Therefore, we have the desired denseness.

(f) The next step is to take any $z \in Z$, let $\{z'_n\}$ be a sequence out of Z' with limit z, and show that lim sup $U_{Z'}(z'_n) =$ lim inf $U_{Z'}(z'_n)$. (The notation here is poor, because the subscript no longer refers to the ennumeration of the previous step.) Suppose this fails. Then we can produce subsequences n' and n'' such that $\lim_{n'} z'_{n'} = z = \lim_{n''} z'_{n''}$, but for all sufficiently large n' and n'', $U_{Z'}(z'_{n'}) > a + \delta$ and $U_{Z'}(z'_{n''}) < a - \delta$ for some a and $\delta > 0$. Use the denseness of $U_{Z'}(Z')$ to produce, out of Z', y' and y'' such that $U_{Z'}(y')$ and $U_{Z'}(y'')$ both come from the interval $(a - \delta/2, a + \delta/2)$, and $y' \succ y''$. Then for all sufficiently large n', $z'_{n'} \succ y'$, and by continuity, $z \succeq y'$. Similarly, for all sufficiently large n'', $y'' \succ z'_{n''}$, so by continuity $y'' \succeq z$. Therefore $z \succeq y' \succ y'' \succeq z$, a contradiction.

Now that we know that $\lim_n U_{Z'}(z'_n)$ exists (if $\lim z_n = z$), we have to show that this limit is finite. We will show that the limit is bounded above by considering two cases. (Showing that the limit is bounded below is symmetric.) (1) If there is a point $x \in X$ such that $V(x) = \bar{v}$, then since we were careful to put one such x in Z', once its $U_{Z'}$ value is assigned this puts an upper bound on $U_{Z'}$. (2) In the other case, where no x is $\succeq$-maximal in x, any $z \in Z$ is strictly worse than some other z^0 in Z, hence (by continuity) it is strictly worse than some z^* in Z'. By continuity again, this must be true for all sufficiently large z'_n, which would bound $U_{Z'}(z'_n)$ (by $U_{Z'}(z^*)$), again giving a finite limit.

(g) This allows us to define $U_Z : Z \to R$ by $U_Z(z) = \lim U_{Z'}(z'_n)$ for any sequence $\{z'_n\}$ out of Z' with limit z. This is clearly well-defined; the limit exists, is finite, and is independent of the sequence $\{z'_n\}$ used to define $U_Z(z)$. (Since Z' is dense in Z, such a sequence always exists.)

To show that U_Z is continuous on Z, let $\{z_n\}$ be a sequence in Z with limit z. By a standard diagonalization argument, we produce a sequence $\{z'_n\}$ from Z' with limit z such that $\lim U_{Z'}(z'_n) = \lim U_Z(z_n)$. (Find $z'_n \in Z'$ such that z'_n is within $1/n$ of z_n and $U_{Z'}(z'_n)$ is within $1/n$ of $U_Z(z_n)$.) Therefore, $\lim U_Z(z_n) = U_Z(z)$.

Finally, we have to show that U_Z represents $\succeq$ on Z. Suppose $z \succ y$ in Z. By continuity of $\succeq$ and denseness of Z' in Z, there are z' and y' in Z' such that $z \succ z' \succ y' \succ y$. We define $U_Z(z)$ by taking a sequence $\{z'_n\}$ out of Z' with limit z and setting $U_Z(z) = \lim U_{Z'}(z'_n)$; by continuity, $z'_n \succ z'$ for all sufficiently large n, thus $U_{Z'}(z'_n) > U_{Z'}(z')$, and hence $U_Z(z) \geq U_{Z'}(z')$. Similarly we can show that $U_{Z'}(y') \geq U_Z(y)$. But since $z' \succ y'$ and $U_{Z'}$ represents $\succeq$ on Z', this implies $U_Z(z) > U_Z(y)$.

Conversely, suppose $U_Z(z) > U_Z(y)$ for z and y from Z. Use the denseness of $U_{Z'}(Z')$ in its range to produce z' and y' such that $U_Z(z) > U_{Z'}(z') > U_{Z'}(y') > U_Z(y)$. Letting $\{z'_n\}$ be a sequence out of Z' with limit z, continuity implies that for all sufficiently large n, $U_{Z'}(z'_n) > U_{Z'}(z')$, thus $z'_n \succ z'$, and therefore, by continuity of $\succeq$, $z \succeq z'$. A similar argument shows that $y' \succeq y$, and hence $z \succ y$. ∎