Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 3: Basics of Consumer Demand

Summary of the Chapter

After the highly abstract Chapter 1 and the moderately abstract Chapter 2, the start of this chapter is likely to come as a relief; it is relatively simple and concrete, and it links back to things you are likely to have studied previously in economics. But by the time the discussion of correspondences and upper semi-continuity is reached, mathematics again takes over.

The chapter does, essentially, three things.

1. The classic consumer's problem of maximizing utility subject to a budget constraint is formulated, and basic properties (homogeneity of the problem in prices and income, existence of a solution, convexity of the solution set, exhaustion of the consumer's budget at the solution) are established.

2. Berge's Theorem, also known as the Theorem of the Maximum, is applied (for the first but far from the last time in this book): As prices and income vary parametrically, the set of solutions forms an upper semi-continuous and locally bounded correspondence, and the value of the solution—the *indirect utility function*—is continuous.

3. Calculus and the first-order/complementary-slackness conditions are used to solve the consumer's problem, and the optimality conditions are interpreted in (what I hope are) fairly intuitive terms.

Do not be misled by the relative shortness of this chapter. It contains three solid topics, and the second and third topics come with fairly tough appendices (Appendices

4 and 5, respectively). Expect this to be a hard chapter to master. In particular, if you are like most students, you will find the material in Appendix 4 to be new and challenging. But if you do master this chapter and its two appendices, a lot of the math that follows in later chapters will come relatively easily. I strongly recommend that you try Problem 3.14 (and, in the Instructor's Manual, I have strongly recommended to your instructor that he or she assign this problem). It isn't about the consumer's problem, *per se*, but if you want to fix the application of ideas from Appendix 4 in your mind, it is an excellent exercise.

Solutions to Starred Problems

■ 3.1. These preferences are locally insatiable everywhere but at the point 0. To see that they are not locally insatiable there, take $\epsilon = 1$. If preferences are locally insatiable at 0, it must be possible to find a consumption bundle $x^* \in R^k_+$ at distance 1 or less from 0 such that $x \succ 0$. But for strictly decreasing preferences, $0 \succeq x$ for all $x \in R^k_+$; indeed, $0 \succ x$ for all $x \in R^k_+$ except for $x = 0$ itself, so no such x^* can be found.

■ 3.4. I solve this using the language of bangs for the buck. The bang for the buck (henceforth, bfb) of goods 1, 2, and 3, respectively, are

$$\frac{1}{2(x_1+2)}, \quad \frac{2}{3(x_2+3)}, \quad \text{and} \quad \frac{4}{x_3+2}.$$

If these are to be equal (which is necessary if the solution has all three commodities strictly positive), we would have to have

$$2(x_1+2) = \frac{3}{2}(x_2+3) = \frac{1}{4}(x_3+2) \quad \text{or} \quad 8x_1 + 16 = 6x_2 + 18 = x_3 + 2.$$

(If a commodity level is zero, its bfb can be smaller than the largest, which means that the corresponding term in the previous display can be larger than the smallest. Since these preferences are locally insatiable (they are strictly monotone), we know that the consumer will spend all her income, or

$$2x_1 + 3x_2 + x_3 = y.$$

At this point, finding the solution for specific values of y is a matter of trial and error. Note that when x_1 is zero, the term corresponding to it in the last string of equalities, $8x_1 + 16$, takes on the value 16. Thus x_1 will be zero until $x_3 + 2 = 16$, or $x_3 = 14$. Similarly, x_2 must be zero until $x_3 + 2 = 18$, or $x_3 = 16$.

Thus when $y = 5$, even if the consumer spends all her wealth on good 3, she only gets $x_3 = 5$ units, and the bfb of good 3 still exceeds the initial bfbs (bangs for the buck) of goods 1 and 2. For $y = 5$, the solution is $x_1 = x_2 = 0$ and $x_3 = 5$.

When $y = 16.4$, the consumer cannot spend all her wealth on good 3: That would give $x_3 = 16.4$, which then requires positive levels of goods 1 and 2. How about spending money on goods 1 and 3 only? We need equal bfbs for these two commodities, or $x_3 + 2 = 8x_1 + 16$, and budget exhaustion, or $2x_1 + x_3 = 16.4$; rewrite the first of these as $x_3 = 8x_1 + 14$ and substitute into the second to get $10x_1 + 14 = 16.4$, or $x_1 = 0.24$, and thus $x_3 = 15.92$. This gives (equal) bfbs for goods 1 and 3 of $1/4.48$, which is larger than $2/9$, the bfb for good 2 at $x_2 = 0$. Therefore, we have our solution: $x_1 = .24, x_2 = 0,$ and $x_3 = 15.92$.

Finally, when $y = 100$, it is easy to guess that all three commodity levels will be strictly positive, so the equal bfbs condition must hold, or $8x_1 + 16 = 6x_2 + 18 = x_3 + 2$. Thus $x_3 = 6x_2 + 16$, and $x_1 = .75x_2 + .25$. Substituting in for x_1 and x_3 in the budget-exhaustion equation gives $2(.75x_2 + .25) + 3x_2 + 6x_2 + 16 = 100$, or $10.5x_2 = 83.5$, and thus $x_2 = 83.5/10.5 = 7.952380\ldots$; this gives $x_1 = 6.2142857\ldots$ and $x_3 = 63.7142857\ldots$.

Suppose we had misguessed in the case $y = 16.4$, guessing instead that the answer would involve all three commodities strictly positive. How would we have learned that this guess is wrong, algebraically? If you run the numbers, you'll find that setting the three bfbs equal, and combining these equations with the budget exhaustion equation, leads to a negative value of x_2, which is the tipoff that this was a bad guess.

■ 3.6. (a) Fixing p, J^* be the set of indices $j = 1, \ldots, k$ that achieve $\max_{j=1,\ldots,k} \alpha_j/p_j$. That is, $j^* \in J^*$ if $\alpha_{j^*}/p_{j^*} \geq \alpha_j/p_j$ for $j = 1, \ldots, k$. Then x^* is a solution of the CP if and only if $p \cdot x^* = y$ and $x^*_j > 0$ only if $j \in J^*$. In words, the consumer must spend all her income, and she must spend it only on commodities that have the highest ratio of α_j to p_j. This is easily derived from the optimality conditions, once you note that $\text{MU}_j \equiv \alpha_j$.

(b) A point x that satisfies the budget constraint $p \cdot x \leq y$ and such that $\alpha_j x_j > \min_i \alpha_i x_i$ cannot be a solution of the CP, since from such a point utility will increase by decreasing x_j by $\epsilon = [\alpha_j x_j - \min_i \alpha_i x_i]/[2\alpha_j]$ (which lowers $\alpha_j x_j$, but not so much that $\alpha_j x_j$ becomes less or equal to $\min_i \alpha_i x_i$), and distributing the resulting income surplus equally over all the other commodities (which rasies the minimum over the other commodities by some strictly positive amount). Therefore, the solution x^* must involve $\alpha_j x^*_j = \alpha_i x^*_i$ for all i and j, or $x^*_j = \alpha_1 x^*_i/\alpha_j$. Thus once x^*_1 is fixed, all other components of x^* are fixed, and $p \cdot x^* = \sum_i p^i[\alpha_1/\alpha_i]x^*_1 = x^*_1 \sum_i p_i[\alpha_1/\alpha_i]$. These preferences are locally insatiable and continuous, so we know that a solution exists and involves $p \cdot x^* = y$, hence

$$x^*_1 = \frac{y}{\alpha_1 \sum_i [p_i/\alpha_i]} \quad ,$$

and hence

$$x^*_j = \frac{\alpha_1}{\alpha_j} x^*_1 = \frac{y}{\alpha_j [\sum_i p_i/\alpha_i]} \quad .$$

■ 3.8. I will describe the procedure, but I am leaving it to you to create the pictures that go with it. I urge you to do so, at least for the example that is worked at as the method is described.

Step 1. For each $i = 1, \ldots, k$, define and draw the function $\lambda_i(y_i) = u_i'(y_i/p_i)/p_i$, where the prime denotes derivative. Note that $u_i'(0)$ is finite, and λ_i has strictly positive values, is continuous, and strictly decreasing. (Continuity follows from the implicit assumption that u_i is continuously differentiable. Strict decreasing follows because u_i is strictly concave. This also implies that the derivatives of the u_i are strictly positive.) The function $\lambda_i(y_i)$ is the *expenditure-driven* bang for the buck of commodity i function. That is, if y_i is spent on commodity i, the amount purchased is y_i/p_i, which has marginal utility $u_i'(y_i/p_i)$, hence bang for the buck equal to $u_i'(y_i/p_i)/p_i$.

Example: Suppose (as in Problem 3.4) that $u(x) = \ln(x_1 + 2) + 2\ln(x_2 + 3) + 4\ln(x_3 + 2)$, and $p = (2, 3, 1)$. Then $\lambda_1(y_1) = (1/(y_1/2 + 2))/2 = 1/(y_1 + 4)$, $\lambda_2(y_2) = (2/(y_2/3 + 3))/3 = 2/(y_2 + 9)$, and $\lambda_3(y_3) = 4/(y_3 + 2)$.

Step 2. Define the function $y_i^* : [0, \infty) \to [0, \infty)$ by letting $y_i^*(\lambda)$ be the unique value that solves the equation $u_i'(y_i^*(\lambda)/p_i)/p_i = \lambda$. If $\lambda > u_i'(0)/p_i$, then set $y_i^*(\lambda) = 0$. If $\lim_{y\to\infty} u_i'(y)/p_i \geq \lambda$, then set $y_i^*(\lambda) = \infty$. (This function y_i^* is the inverse of the function graphed in Step 1, so if we graph it with λ on the vertical axis, we get the "same" graph as in Step 1.)

Example: Since $\lambda_1(y_1) = 1/(y_1 + 4)$, we solve $y_1^*(\lambda) = 1/\lambda - 4$ (for $\lambda < 1/4$). Similarly, $y_2^*(\lambda) = 2/\lambda - 9$ (for $\lambda < 2/9$), and $y_3^*(\lambda) = 4/\lambda - 2$ (for $\lambda < 2$).

Step 3. Let $y^*(\lambda) = \sum_{i=1}^k y_i^*(\lambda)$. That is, $y^*(\lambda)$ is the horizontal sum of the y_I^* functions (horizontal because we put λ on the vertical axis). I assert that $y^*(\cdot)$ is strictly decreasing where it is finite and nonzero, continuous, and satisfies $\lim_{\lambda\to 0} y^*(\lambda) = \infty$. I leave the first two pieces of this assertion to your ingenuity and only indicate why the last is true: For any y, if $\lambda < u_i'(y/p_1)/p_1$, then $y^*(\lambda) > y$.

Example: We have

$$y^*(\lambda) = \begin{cases} 0, & \text{if } \lambda \geq 2, \\ 4/\lambda - 2, & \text{if } 2 > \lambda \geq 1/4, \\ 5/\lambda - 6, & \text{if } 1/4 > \lambda \geq 2/9, \text{ and} \\ 7/\lambda - 15, & \text{if } 2/9 > \lambda. \end{cases}$$

Step 4. Take the fixed income level $y > 0$ and find the unique value of λ such that $y^*(\lambda) = y$. A unique value of λ solves this equation because y^* is continuous and approaches ∞ as λ approaches zero (existence) and is strictly decreasing where it is positive (uniqueness). Then the solution to the CP for p and y is x^*, where $x_i^* = y_i^*(\lambda)/p_i$.

For the utility function of Problem 3.4, we have described the corresponding y^* function in the display in Step 3. Note the values of y at the critical values of λ: $y^*(2) = 0$; $y^*(1/4) = 14$; $y^*(2/9) = 33/2 = 16.5$. So for, say, $y = 100$, we know that the corresponding λ value must be in the range from 2/9 to 0, hence must be a solution of $7/\lambda - 15 = 100$, or $\lambda = 7/115$. This gives $y_1^*(7/115) = 115/7 - 4 = 12.4285\ldots$, hence x_1 (at the solution for $y = 100$) is $(12.4285\ldots)/2 = 6.2142\ldots$. (Compare this with the answer obtained in Problem 3.4.)

Why does this produce a solution? We know (because the u_i are strictly concave) that there is a unique solution to the problem, and the optimality conditions are necessary and sufficient for the solution. Then fixing p and y, simply check that this procedure generates a solution to the optimality conditions: The optimality conditions are $u_i'(x_i^*) = p_i\lambda$ for x_i^* that are strictly positive, and $u_i'(0) \leq p_i\lambda$ if $x_i^* = 0$, plus full expenditure, which is precisely the set of conditions generated by the procedure.

■ 3.9. Fix p and y. For each $i = 1, \ldots, k-1$, let $\hat{x}_i = 0$ if $u_i'(0) < p_i/p_k$ and otherwise satisfy $u_i'(\hat{x}_i) = p_i/p_k$. If $\sum_{i=1}^{k-1} p_i\hat{x}_i \leq y$, then the solution is $x_i^* = \hat{x}_i$ for $i = 1, \ldots, k-1$ with the remainder of y "spent" on the kth good, while if $\sum_{i=1}^{k-1} p_i\hat{x}_i > y$, then the solution is found by setting $x_k^* = 0$ and using the graphical procedure of Problem 3.8 for the first $k-1$ components.

I'll leave it to you to figure out why this works. (Of course, the reason is that this guarantees a solution to the optimality conditions, and your job is to verify that this is so.) But one comment is definitely in order: This formulation for utility — quasi-linear in a single commodity — is quite common in applications, where the commodity is "money left over," and the interpretation is that the $k-1$ goods are only a partial list of what the consumer will buy. Note that in our solution, hence in such applications, as long as the consumer has enough money in the sense that $y \geq \sum_{i=1}^{k-1} p_i\hat{x}_i$, the consumer's choice of the $k-1$ initial goods is independent of y; she buys good i to the point where its bang for the buck is $1/p_k$, which is the unchanging bang for the buck of the kth commodity.

■ 3.12. Let $k = 1$, $u(x_1) = (x_1 - 4)^3$, $p = (1)$, and $y = 4$. I should explain what $p = (1)$ means: there is a single commodity, so $p = (p_1)$, and $p_1 = 1$. Preferences are continuous (since u is), strictly increasing (hence, locally insatiable), and strictly convex (how do I know this?), so we know that at the solution to the CP, $p \cdot x_1^* = y = 4$, which means that $x_1^* = 4$. But $\text{MU}_1 = 3(x_1 - 4)^2$, which at $x_1^* = 4$ is zero, so the optimality conditions $\text{MU}_1(x_1^*)/p_1 = \lambda$ give $\lambda = 0$.