Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 9: Competitive and Profit-Maximizing Firms

Summary of the Chapter

In the economics of this book, two types of economic entities interact: consumers and firms. Chapters 1 through 8 have concerned consumers, as will Chapters 10 and 11. In this chapter, firms are studied. (Subsequent chapters concern both consumers and firms.)

Firms are entities that have the capability of transforming a vector of commodities, the inputs to its production process, into another vector of commodities, its outputs. This is done, in this chapter at least, instantaneously. The model of a firm specifies two things:

1. Its technological capabilities: Which vectors of inputs can the firm transform, into which vectors of outputs?
2. What does the firm choose to do, given its various technological capabilities?

For almost all of this chapter, the firm's capabilities are modeled with a *production-possibility set*, discussed in Section 9.1. Its objective, which guides its choice of production plan given prices, is *profit maximization*, discussed in Section 9.2. Having put these two pieces of the model of the firm in place, the bulk of the chapter analyzes the firm's profit-maximization problem. Section 9.3 concerns the existence of solutions to the firm's profit-maximization problem—nontrivial because production-possibility sets are not compact in general—and basic properties of solutions. In particular, we show that, viewed as a correspondence whose domain is the space of strictly positive prices, the firm's optimal netput set is upper semi-continuous; viewed as a function with the same domain, the firm's profit function is convex, homogeneous of degree 1, and continuous. (Of course, certain assumptions about the production-possibility set are required to prove these results.) In Section 9.4, we derive results similar in character to Afriat's Theorem, for firms instead of for the market demand of a consumer.

Through Section 9.4, the analysis parallels the analysis of the consumer's utility-maximization problem in Chapters 3 and 4. Section 9.5 breaks new analytical ground. The focus is on the firm's *profit function*, which says how much profit the firm can (optimally) make as function of the prices it faces. In particular, we study in depth the question of *duality*: To what extent does the firm's profit function identify its production-possibility set?

1. Section 9.5 shows that if a function $\pi : R^k_{++} \to R$ is convex, homogeneous of degree 1, and continuous (which it must be if it is convex), then we can identify a production-possibility set that has π as its profit function. This set is convex, closed, and has free-disposal (meaning that with the same level or more inputs, the firm can always make the same level or less output).

2. More than one production-possibility set can produce the same profit function. Section 9.6 answers the questions, When do two production-possibility sets give the same profit function? When do they give different profit functions?

3. The results of Sections 9.5 and 9.6 establish a *duality* or one-to-one correspondence between profit functions (functions with domain R^k_{++} and range R) and production-possibility sets that are closed, convex, and have free disposal. Section 9.7 looks harder at the mathematics of this duality result, tieing it (in general) to the mathematical result that a convex set in R^k that is less than all of R^k is the intersection of the half-spaces that contain it, the so-called *Support-Function Theorem*.

4. In Section 9.8, we show that a profit function is differentiable at prices p^0 if and only if the firm's profit-maximization problem admits a single solution at p^0.

A different approach to modeling the firm involves *input-requirement sets* and the firm's cost-minimization problem. This approach is briefly outlined in Section 9.9.

What this chapter does not discuss

Throughout Chapter 9, the firm is *competitive* or a *price taker*, meaning that it acts in the belief that its choice of production plan will have no impact on the price of any commodity. While this may be a reasonable assumption for (most) consumers, it is less reasonable when it comes to firms and especially with regard to the outputs of firms. For instance, General Motors probably does not take the price of Cadillacs as given. (One reason to study the firm's cost-minimization theory is that this assumes that the firm is a price taker [only] for inputs to its production process, a generally more palatable assumption.) Theories of monopolistic firms, oligopolistic firms, and monopolistically competitive firms all address the "firm as entity with given production capabilities and with profit-maximizing objectives," but with some effect on prices. We do not study any of this here, leaving it for Volume II.

In some treatments of the firm in the literature, the firm has capabilities specified as here and may or may not have the ability to affect the prices it faces, but it does not seek maximal profits. Instead, it chooses its production plan in pursuit of other objectives, such as the maximization of its wage bill in so-called worker-managed firms or the maximization of its fixed asset base in some managerial theories of the firm. We do not discuss such models at all.

And, especially in the recent past, a lot of attention has been paid to thinking of firms not as entities with a given set of capabilities and a fixed objective but instead as an institution much akin to a market, within which various entities—workers, owners (shareholders), managers, customers, suppliers—interact. The study of firms as settings for the interaction of various parties, rather than as entities, is left in part to Volume II and especially to Volume III.

Solutions to Starred Problems

■ 9.2. Suppose $0 \in Z$. Then it is clear that for every price $p \in R^k_{++}$, $\pi(p) \geq 0$. And if $\pi(p) \geq 0$ for all p, then 0 is obviously in the set

$$\{z \in R^k : p \cdot z \leq \pi(p) \text{ for all strictly positive } p\}.$$

So if π is nonnegative, then while 0 may not be in Z, 0 is in the closed free-disposal convex hull of Z. (Since you were told to assume that π is real-valued, Corollary 9.18 and Proposition 9.16 apply.)

■ 9.4. Yes, $p_1 \to z_1(p_1)$ is continuous in this open domain. But to show this takes a bit of work. Corollary A4.10 says that if $p \Rightarrow Z^*(p)$ is singleton valued in an open domain, the function described is a continuous function, hence the projection of this function along any coordinate describes a continuous function. This is a corollary of Berge's Theorem. What is different here is that while $Z^*(p)$ may not be singleton valued, its projection along the first coordinate is singleton valued. But the proof mimics the proof from Berge's Theorem:

Take any sequence $\{p_1^n\}$ of prices for the first coordinate, lying within this open set and with limit p_1^0. We will show that $\lim_n z_1(p_1^n) = z_1(p_1^0)$, which demonstrates continuity of the $z_1(\cdot)$ function.

Let $p^n = (p_1^n, p_2, \ldots, p_k)$, and let z^n be any element of $Z^*(p^n)$. We know that the first coordinate of z^n, z_1^n, is $z_1(p_1^n)$. Because Z has the recession-cone property, we know from Lemma 9.8 that the sequence $\{z^n\}$ lives within a bounded set, so by looking at a subsequence if necessary, we know that $\{z^n\}$ has a limit z^0. By Berge's Theorem, $z^0 \in Z^*(p_1^0, p_2, \ldots, p_k)$ and, therefore, $z_1^0 = z_1(p_1^0)$. But this implies that $z_1(p_1^0)$ is indeed the limit of $\{z_1^n\} = \{z_1(p_1^n)\}$. That does it.

■ 9.7. Suppose that $k = 2$, $Z = \{(x_1, x_2) : x_1 \leq 0\}$, $Z' = \{(x_1, x_2) : x_1 \leq -1\}$, and $Z'' = \{(x_1, x_2) : x_2 \leq -10\}$. In words, Z represents a technology which can get as much of the second good as desired with no inputs, Z' represents a technology which can get as much of the second good as desired if there is at least 1 unit of good 1 input, and Z'' represents a technology which can get as much of the first good as desired if there is at least 10 units of the second good input. Each of these is closed, convex, and has free disposal, and they are certainly distinct from one another. But at any strictly positive prices, each of them gives a profit of ∞, so they share the same ($\equiv \infty$) profit function (for strictly positive prices).

They have different profit functions if we can have zero prices. At the price vector $(1, 0)$, Z gives profit 0, Z' gives profit -1, and Z'' gives profit ∞, while at $(0, 1)$, Z and Z' both give profit ∞ while Z'' gives profit -10.

Now for the proof of the corollary: I'll show that first that, for any $Z \subseteq R^k$, Z and $\overline{\text{FDCH}}(Z)$ give the same profit for any nonnegative p. Therefore, if $\overline{\text{FDCH}}(Z) = \overline{\text{FDCH}}(Z')$, Z and Z' have the same extended profit function (which is the same as the extended profit function for their common $\overline{\text{FDCH}}$s.

Look at the proof of Proposition 9.16, starting with the second paragraph. No changes are necessary in the second paragraph. In the third paragraph, change "for some strictly positive price p" to "for some nonnegative price p" in the second sentence. Change "(since prices are strictly positive)" to "(since prices are nonnegative)" in the sixth sentence. With these changes, the argument goes through without a hitch.

For the converse, since we know that Z and $\overline{\text{FDCH}}(Z)$ have the same profit function and since Z' and $\overline{\text{FDCH}}(Z')$ have the same profit function, I have the result if I show that, *if Z and Z' are distinct closed, convex, and free-disposal subsets of R^k, then they have different extended profit functions.* This time, it is easier just to write down the argument.

Since Z and Z' are distinct, either there is some $z \in Z \setminus Z'$ or some $z' \in Z' \setminus Z$ (or both). Wlog, assume there is some $z \in Z \setminus Z'$. Then since Z' is convex and closed, we can find a hyperplane that strictly separates the point z from Z'; that is, for some $p \in R^k$ that is not identically 0, $p \cdot z > \sup_{z' \in Z'} p \cdot z'$. This implies, in particular, that $\sup_{z' \in Z'} p \cdot z' < \infty$, which implies that $p \geq 0$: If some component of p were strictly less than zero, then take any z^0 from Z'[1] and then look at $z^M = z^0 + (0, 0, \ldots, 0, -M, 0, \ldots, 0)$, where the $-M$ goes in the coordinate for which p is strictly negative. As M goes to ∞, $p \cdot z^M$ goes to ∞. But $z^M \in Z'$ by free disposal, which contradicts that $\sup_{z' \in Z'} p \cdot z' < \infty$.

But then $\overline{\pi}_Z(p) \geq p \cdot z > \sup_{z' \in Z'} p \cdot z' = \overline{\pi}_{Z'}(p)$, showing that Z and Z' have distinct extended profit functions.

■ 9.9. (a) First I'll show that if $Z' = \overline{\text{CH}}(Z)$, then $\tilde{\pi}_Z \equiv \tilde{\pi}_{Z'}$. This will immediately establish that if $\overline{\text{CH}}(Z) = \overline{\text{CH}}(Z')$, then $\tilde{\pi}_Z \equiv \tilde{\pi}_{Z'}$ in general.

So suppose that $Z' = \overline{\text{CH}}(Z)$. Since $Z \subseteq Z'$ and since the support functions are defined as suprema, it is clear that $\tilde{\pi}_{Z'} \geq \tilde{\pi}_Z$. Suppose by way of contradiction that $\tilde{\pi}_{Z'}(p) > \tilde{\pi}_Z(p)$ for some $p \in R^k$. Then for some $\epsilon > 0$, we can find $\hat{z} \in \overline{\text{CH}}(Z)$ such

[1] If $Z' = \emptyset$, then $\overline{\pi}_{Z'} \equiv -\infty$ which is different from $\overline{\pi}_Z$, since we know that Z is not empty.

that $p \cdot \hat{z} > \tilde{\pi}_Z(p) + \epsilon$. By continuity of the dot product, we can find some $z' \in \mathrm{CH}(Z)$ close enough to z so that $p \cdot z' > \tilde{\pi}_Z(p) + \epsilon/2$. Since $z' \in \mathrm{CH}(Z)$, z' is a convex combination of elements of Z; write $z' = \sum_{j=1}^n \alpha^j z^j$, where each $z^j \in Z$ and the scalars are nonnegative and sum to one. Since $p \cdot z' = \sum_{j=1}^n \alpha^j (p \cdot z^j) \geq \tilde{\pi}_Z(p) + \epsilon/2$, and since the α^j are nonnegative and sum to one, for some index j, $p \cdot z^j \geq \tilde{\pi}(p) + \epsilon/2$. Since this z^j is from Z, we have a contradiction.

For the converse, suppose $\overline{\mathrm{CH}}(Z) \neq \overline{\mathrm{CH}}(Z')$. Since the support functions of Z and of $\overline{\mathrm{CH}}(Z)$ are identical, as are those of Z' and $\overline{\mathrm{CH}}(Z')$, we need to show that the support functions of $\overline{\mathrm{CH}}(Z)$ and $\overline{\mathrm{CH}}(Z')$ are not the same. This amounts to showing that if Z and Z' are different closed and convex sets, they have different support functions, which is what I'll do.

So suppose Z and Z' are different closed and convex sets. Either $Z \setminus Z'$ or $Z' \setminus Z$ (or both) is nonempty. So, wlog, suppose that $z \in Z \setminus Z'$. Use the Strict-Separation Theorem to find a $p \in R^k$, not identically zero, such that $p \cdot z > \sup\{p \cdot z' : z' \in Z' = \tilde{\pi}_{Z'}(p)$. Since $\tilde{\pi}_Z(p) \geq p \cdot z$ for this z (since $z \in Z$), we're done.

(b) If Z is compact, it is obvious that, for every $p \in R^k$, $\tilde{\pi}_Z(p) = \sup\{p \cdot z : z \in Z\} < \infty$, since the dot product is a continuous function in z for every fixed p. Conversely, suppose Z is closed and convex and that, for every p, $\tilde{\pi}_Z(p) = \sup\{p \cdot z : z \in Z\} < \infty$. Suppose by way of contradiction that Z is not bounded. Then we can find a sequence $\{z^n\}$ from Z such that $\|z^n\| > n$. Let $\hat{z}^n = z^n/\|z^n\|$; $\{\hat{z}^n\}$ is a sequence that lies on the unique sphere, hence is has an accumulation point $\hat{z}^0$ lying on the unit sphere. $\hat{z}^0$ is not identically zero, so it has some nonzero component, say component i; let $e_i = 1$ or -1 depending on whether $\hat{z}_i^0$ is positive or negative, and let $p = (0, \ldots, 0, e_i, 0, \ldots, 0)$ where e_i is put in coordinate i. Then $p \cdot \hat{z}^0 = |z_i^0| > 0$. Therefore, by continuity, $p \cdot \hat{z}^n \to |z_i^0|$ as $n \to \infty$ along the subsequence for which $\hat{z}^n$ converges to $\hat{z}^0$. But then $p \cdot z^n = \|z^n\| p \cdot \hat{z}^n > np \cdot \hat{z}^n$ when n is large enough in the subsequence so that $p \cdot \hat{z}^n$ has become positive, and therefore $\lim_n p \cdot z^n$ along that subsequence is $+\infty$. This means that $\tilde{\pi}_Z(p) = \infty$ for this p, a contradiction.

Now we must show that $\tilde{\pi}_Z$ is convex and homogeneous of degree 1. If it is not convex, then there exist p, p', and $\alpha \in [0, 1]$ such that $\alpha\tilde{\pi}_Z(p) + (1 - \alpha)\tilde{\pi}_Z(p') < \tilde{\pi}_Z(\alpha p + (1 - \alpha)p')$. For this to be true, there must exist $z \in Z$ such that $\alpha\tilde{\pi}_Z(p) + (1 - \alpha)\tilde{\pi}_Z(p') < (\alpha p + (1 - \alpha)p') \cdot z = \alpha(p \cdot z) + (1 - \alpha)(p' \cdot z) \leq \alpha\tilde{\pi}_Z(p) + (1 - \alpha)\tilde{\pi}_Z(p')$, which is a contradiction. (You should be careful that everything I said works, even if $\tilde{\pi}_Z$ can be infinite valued. But it does work; the key is the very first inequality: Convexity fails only if $\alpha\tilde{\pi}_Z(p) + (1 - \alpha)\tilde{\pi}_Z(p') < \tilde{\pi}_Z(\alpha p + (1 - \alpha)p')$ for some p, p', and α. This can't happen for $\alpha = 0$ or 1, since then the two sides are identical, therefore we can assume $\alpha \in (0, 1)$. But then if the right-hand side is finite, so is the left-hand side (since it is less); and if the right-hand side is infinite, the left-hand side to be less must be finite. Once you see that, it all goes through very nicely.) And for homogeneity of degree one, this fails only if for some p and some $\alpha \geq 0$, $\tilde{\pi}_Z(\alpha p) \neq \alpha\tilde{\pi}_Z(p)$. If $\alpha = 0$, this can't happen by definition (since $\tilde{\pi}_Z(0) = 0$), so we can assume $\alpha > 0$. And then by perhaps replacing α with $1/\alpha$ and p with αp, we can assume wlog that we have $\alpha > 0$ and p such that $\tilde{\pi}_Z(\alpha p) > \alpha\tilde{\pi}_Z(p)$. But then there must exist $z \in Z$ such that

$\alpha\tilde{\pi}_Z(p) < (\alpha p) \cdot z = \alpha(p \cdot z) \leq \alpha\tilde{\pi}_Z(p)$, a contradiction.

If you compare these proofs of convexity and homogeneity with the proofs used for the profit function when we assumed it was everywhere finite valued, you will see that they are essentially identical.

(c) We are now assuming that we have a convex and homogeneous real-valued function $\tilde{\pi} : R^k \to R$, for which we define

$$Z = \{z \in R^k : p \cdot z \leq \tilde{\pi}(p) \text{ for all } p \in R^k\}.$$

If we then define

$$\tilde{\pi}_Z(p) = \sup\{p \cdot z : z \in Z\},$$

it is immediate that $\tilde{\pi}_Z(p) \leq \tilde{\pi}(p)$, since for each p, if $z \in Z$, then $p \cdot z \leq \tilde{\pi}(p)$, so taking the supremum of these values over all $z \in Z$ can't get us above this common upper bound.

The key, then, is to show that for each $p \in R^k$, there is some $z_p \in Z$ such that $p \cdot z_p = \tilde{\pi}(p)$. But this is easy: Fix p. Since $\tilde{\pi}$ is convex and since its domain is open (so that p is in the interior of the domain), there is a subgradient of $\tilde{\pi}$ at p. Because $\tilde{\pi}$ is homogeneous of degree 1, subgradients take the form of scalar multiplication by a non-zero vector (which I'll call) $z_p \in R^k$; that is, $p \cdot z_p = \tilde{\pi}(p)$ and $p' \cdot z_p \leq \tilde{\pi}(p')$ for all p'. Therefore, $z_p \in Z$. And $p \cdot z_p = \tilde{\pi}(p)$ implies $\tilde{\pi}_Z(p) \geq \tilde{\pi}(p)$. Combined with our earlier inequality (the other way), this gives us the result.

■ 9.11. (a) and (b) are trivial matters of comparing two ways to say the same things.

(c) If $x \in V(y)$, then $(y, -x, 0) \in Z$. If $x' \geq x$, then $-x' \leq -x$, so $(y, -x', 0) \leq (y, -x, 0)$, so $(y, -x', 0) \in Z$ by free disposal, and therefore $x' \in V(y)$.

See Figure G9.1(a) for an example where each $V(y)$ is comprehensive upwards, but Z does not have free disposal.

Suppose $\hat{Z}$ has free disposal. By the argument just given, this implies that each $V(y)$ is comprehensive upwards. And if $y \geq y'$ and $x \in V(y)$, then $(y, -x, 0) \in X$, hence (since $y \geq y'$ and $\hat{Z}$ has free disposal) $(y', -x, 0) \in \hat{Z}$, hence $x \in V(y')$, and $V(y) \subseteq V(y')$; the V sets nest.

In the other direction, suppose the $V(y)$ are comprehensive upwards and nest, and Y is "comprehensive downwards." If $(y, -x, 0) \in \hat{Z}$ and $(y', -x', 0) \leq (y, -x, 0)$, then $y \geq y' \geq 0$ and $x' \geq x$. Since $(y, -x, 0) \in \hat{Z}, x \in V(y)$. Since $y \geq y'$ and the V's nest, $y' \in Y$ and $x \in V(y) \subseteq V(y')$, hence $x \in V(y')$. Since the Vs are comprehensive upwards, $x' \in V(y')$. Therefore $(y', -x', 0) \in V(y')$; we have (modified for $\hat{Z}$) free disposal.

(d) Suppose $x, x' \in V(y)$. This implies $(y, -x, 0) \in Z$ and $(y, -x', 0) \in Z$. Therefore, for all $\alpha \in [0, 1]$, $(y, -(\alpha x + (1 - \alpha)x'), 0) \in Z$, which implies $\alpha x + (1 - \alpha)x' \in V(y)$.

Figure G9.1(a) is also a case where each $V(y)$ is convex, but $\hat{Z}$ (and Z) is not.

Proving the last part of d just amounts to rearranging definitions.

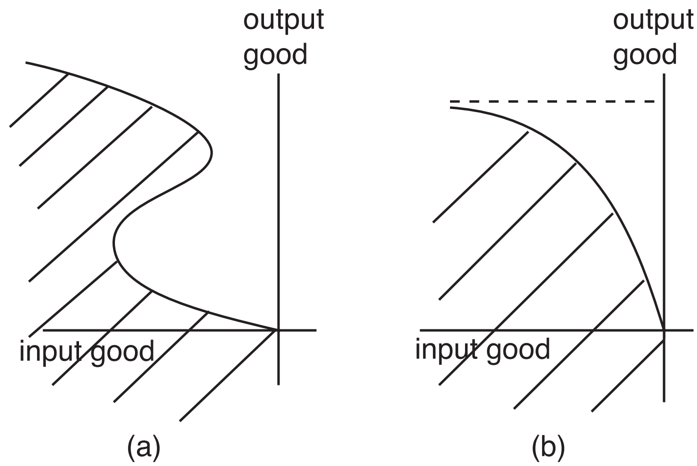

Fig. G9.1. Two Production-Possibility Sets. The two production-possibility sets depicted provide "counterexamples" for Problem 9.11. In each case, there is a single output good and a single input good, and the whole production possibility set Z is shown as the hatched area, with free disposal assumed. In panel a, each input requirement set $V(y)$ is convex and comprehensive upwards, but Z is neither convex not does it have free disposal. In panel b, the boundary of Z is meant to asymptote to the dashed line but never hit it; so Y includes all levels of output up to but not including the level of the dashed line; although Z is closed, Y is not.

(e) Suppose $y \Rightarrow V(y)$ is upper semi-continuous. Suppose that $(y^\ell, -x^\ell, 0)$ is a sequence from $\hat{Z}$ such that y^ℓ converges to y and $-x^\ell$ converges to some $-x$. Since we are assuming this sequence is drawn from $\hat{Z}$, this means that $x^\ell \in V(y^\ell)$ for each ℓ, and then upper semi-continuity of the V correspondence tells us that $x \in V(y)$, which implies that $(y, -x, 0) \in \hat{Z}$, and $\hat{Z}$ is closed.

Conversely, suppose that $\hat{Z}$ is closed and that we have sequences $\{y^\ell\}$ and $\{x^\ell\}$ with respective limits y and x and with $x^\ell \in V(y^\ell)$ for each ℓ. The last piece of this implies that $(y^\ell, -x^\ell, 0) \in \hat{Z}$ for all ℓ, and so the closedness of $\hat{Z}$ implies $(y, -x, 0)$ in $\hat{Z}$, which immediately yields $x \in V(y)$.

(Or you can subsume both paragraphs into the citation of Proposition A4.3, if you note the connection between $\hat{Z}$ and the graph of the V correspondence.)

The corollary statement, that if $\hat{Z}$ is closed, then each $V(y)$ is closed, follows immediately from Proposition A4.4.

But we cannot guarantee that Y is closed; see panel b of Figure G9.1.

▪ 9.12. The proof of part a is obvious: $r \in R^n_{++}$ and $x \in R^n_+$, so the function over which we are minimizing, $r \cdot x$, is always nonnegative.

For part b, let x^0 be an arbitrary element of $V(y)$. Then any solution to this problem x^* must satisfy $r \cdot x^* \leq r \cdot x^0$. So we can restrict attention in the problem to the set

$$\{x \in V(y) : r \cdot x \leq r \cdot x^0\}.$$

But if $V(y)$ is closed, this set is compact: As the intersection of two closed sets, namely $V(y)$ and $\{x : r \cdot x \leq r \cdot x^0\}$, it is itself closed. And as r is strictly positive, the set in the display just above is strictly bounded, since $\{x : r \cdot x \leq r \cdot x^0\}$ is bounded: See the proof of Proposition 3.1(b).

Part c follows the usual lines: The objective function (being minimized) is linear, hence convex in x, and the constraint set is (by assumption) convex.

For part d, take any two vectors of factor prices r and r' and $\alpha \in (0, 1)$. Let $r'' = \alpha r + (1 - \alpha)r'$, and let $x^n \in V(y)$ be such that $r'' \cdot x^n \leq \mathrm{C}(r'') + 1/n$. Then

$$\alpha\mathrm{C}(r) + (1 - \alpha)\mathrm{C}(r') \leq \alpha r \cdot x^n + (1 - \alpha)r' \cdot x^n = (\alpha r + (1 - \alpha r') \cdot x^n = r'' \cdot x^n \leq \mathrm{C}(r'') + 1/n.$$

Since n is arbitrary here, this implies $\alpha\mathrm{C}(r) + (1 - \alpha)\mathrm{C}(r') \leq \mathrm{C}(r'')$, and C is concave.

Part e: Fix $r \in R^n_{++}$ and $\alpha > 0$. Suppose that $x^n \in V(y)$ is such that $r \cdot x^n \leq \mathrm{C}(r) + 1/n$. Then $\mathrm{C}(\alpha r) \leq (\alpha r) \cdot x^n = \alpha(r \cdot x^n) \leq \alpha\mathrm{C}(r) + \alpha/n$. Since n is arbitrary here, $\mathrm{C}(\alpha r) \leq \alpha\mathrm{C}(r)$. But if we replace α with $1/\alpha$ and r with αr in the inequality just derived, we get $\mathrm{C}(r) = \mathrm{C}((1/\alpha)\alpha r) \leq (1/\alpha)\mathrm{C}(\alpha r)$, and multiplying through both sides of this inequality by α gives $\alpha\mathrm{C}(r) \leq \mathrm{C}(\alpha r)$. Therefore, $\alpha\mathrm{C}(r) = \mathrm{C}(\alpha r)$.

Part f: If $x \in V^*(r)$, then $r \cdot x = \mathrm{C}(r)$ and $x \in V(y)$, hence $r' \cdot x \geq \mathrm{C}(r')$ for all r'. That makes x a supergradient of C at r.

Conversely, if x is in $V(y)$ and x is a supergradient of C at r, then for some scalar β, $r \cdot x + \beta = \mathrm{C}(r)$, and $r' \cdot x + \beta \geq \mathrm{C}(r')$ for all other r'. In particular, this is true for $r' = \alpha r$, for $\alpha > 0$, so that $\alpha r \cdot x + \beta \geq \mathrm{C}(\alpha r) = \alpha\mathrm{C}(r) = \alpha(r \cdot x + \beta) = \alpha r \cdot x + \alpha\beta$. Cancelling the term $\alpha r \cdot x$ from both sides of this inequality, we get $\beta \geq \alpha\beta$ for all $\alpha > 0$, which implies $\beta = 0$. Therefore, $r \cdot x = \mathrm{C}(r)$, which means that $x \in V^*(r)$.

That takes care of the easy parts of the proposition. Next comes part g, or *Berge's Theorem* (for fixed y and varying r):

Having fixed y, the problem is to minimize $r \cdot x$ subject to $x \in V(y)$. The objective function $r \cdot x$ is jointly continuous in r and x, because it is bilinear in the two arguments. The constraint correspondence $r \Rightarrow V(y)$ is constant in r, so it clearly is lower semi-continuous. So the only challenge here it to find the locally bounded, upper semi-continuous sub-correspondence within which all the solutions can be found. Let x^0 be any element of $V(y)$, and let $B(r) = \{x \in V(y) : r \cdot x \leq r \cdot x^0\}$. It is then clear minimizing $r \cdot x$ over $B(r)$ gives the same infimum as does minimizing over $V(y)$ and that any solution to the cost-minimization problem at prices r will be contained within $B(r)$ (for the fixed y), since any solution must have a total cost no greater than the cost of obtaining y using the (feasible) input set x^0. So we're done once we show that $r \Rightarrow B(r)$ is locally bounded and upper semi-continuous.

For local boundedness, fix $r^0 \in R^n_{++}$. Let $K = r^0 \cdot x^0$, $M = \max\{x^0_i; i = 1, \ldots, n\}$, and $\delta = \min\{r^0_i; i = 1, \ldots, n\}$. I assert that if $\|r - r^0\| \leq \delta/2$, then for all $x \in B(r)$, $\|x\| < 2n^2(K + M\delta/2)/\delta$. To see this, first note that if $\|r - r^0\| \leq \delta/2$, then by the triangle inequality, $|r_i - r^0_i| \leq \delta/2$ and therefore $r_i \geq \delta/2$. The first inequality tells us that $|r \cdot x^0 - r^0 \cdot x^0| \leq n \max\{|r_i - r^0_i|x^0_i; i = 1, \ldots, n\} \leq n(\delta/2)M$, therefore $r \cdot x^0 \leq$

$n(\delta/2)M + K \leq n(K + M\delta/2)$. So for x to be in $B(r)$, we need that $r \cdot x \leq n(K + M\delta/2)$. But since all the components of r are strictly positive and all the components of x are nonnegative, this implies that for each $i = 1, \ldots, n$, $r_i x_i \leq n(K + M\delta/2)$. And since $r_i \geq \delta/2$, this in turn implies that $x_i \leq n(K + M\delta/2)/(\delta/2) = 2n(K + M\delta/2)/\delta$. This then gives the upper bound on $\|x\|$ that I asserted.

And for upper semi-continuity of $r \Rightarrow B(r)$, suppose $r^n \to r$, $x^n \to x$, and $x^n \in B(r^n)$ for each n. The last part of this says that $x^n \in V(y)$ and $r^n \cdot x^n \leq r^n \cdot x^0$ for each n. Since $V(y)$ is closed, $x \in V(y)$, and by continuity of the dot product, taking the limit on both sides of the inequalities $r^n \cdot x^n \leq r^n \cdot x^0$ tells us that $r \cdot x \leq r \cdot x^0$, which implies that $x \in B(r)$.

Part h:

Fix a set $X \subseteq R^n_+$. Let C be the "cost function" for X—that is, $\mathrm{C} : R^n_{++} \to R$ is defined by $\mathrm{C}(r) = \inf\{r \cdot x : x \in X\}$—and let C′ be the cost function for $\overline{\mathrm{CCH}}(X)$. Since $X \subseteq \overline{\mathrm{CCH}}(X)$ and the two functions are defined using infima, we know that $\mathrm{C}'(r) \leq \mathrm{C}(r)$ for all r. Moreover, C and C′ are both nonnegative. So once we show that $\mathrm{C}'(r) < \mathrm{C}(r)$ is impossible, we'll know that $\mathrm{C} \equiv \mathrm{C}'$, proving the first half of the result.

Suppose $\mathrm{C}'(r) < \mathrm{C}(r)$ for some specific r. The value of $\mathrm{C}'(r)$ is gotten by taking the infimum of $r \cdot x$ for x in the closure of CCH(X), so (since the dot product is continuous) there must be some $x \in \mathrm{CCH}(X)$ such that $r \cdot x < \mathrm{C}(r)$. But for x to be in CCH(X), it must be that $x \geq \sum_{\ell}, \alpha_\ell x^\ell$ where the α's are nonnegative scalars that sum to one and each x^ℓ is in X. Since factor prices (components of r) are strictly positive, $r \cdot x \geq \sum_\ell \alpha_\ell (r \cdot x^\ell)$, therefore $\sum_\ell \alpha_\ell (r \cdot x^\ell) < \mathrm{C}(r)$. But since $x^\ell \in X$ for all ℓ, $r \cdot x^\ell \geq \mathrm{C}(r)$ for all ℓ, which gives us $\mathrm{C}(r) = \sum_\ell \alpha_\ell \mathrm{C}(r) \leq \sum_\ell \alpha_\ell (r \cdot x^\ell) < \mathrm{C}(r)$, a contradiction.

For the converse, fix X and X' such that $\overline{\mathrm{CCH}}(X) \neq \overline{\mathrm{CCH}}(X')$. Without loss of generality, suppose $x^0 \in \overline{\mathrm{CCH}}(X) \setminus \overline{\mathrm{CCH}}(X')$. Since $\overline{\mathrm{CCH}}(X')$ is closed and convex, the Strict-Separation Theorem tells us that there exist $r^0 \in R^n$ (no sign restriction, yet) such that $r^0 \cdot x^0 < \inf\{r^0 \cdot x : x \in \overline{\mathrm{CCH}}(X')\}$. Since $\overline{\mathrm{CCH}}(X')$ is comprehensive upwards, it is easy to see that $r^0 \geq 0$. Now let e denote the vector $(1, 1, \ldots, 1) \in R^n$, and let $c = e \cdot x^0$. Let ϵ be small enough (but strictly positive) so that $r^0 \cdot x^0 + \epsilon c < \inf\{r^0 \cdot x : x \in \overline{\mathrm{CCH}}(X')\}$. Then if $r = r^0 + \epsilon e$, which is strictly positive, $r \cdot x^0 < \inf\{r \cdot x : x \in \overline{\mathrm{CCH}}(X')\}$ (since $r \cdot x \geq r^0 \cdot x$ for all nonnegative x), and therefore the cost function associated with X, evaluated at the strictly positive price vector r, will give a value less than the cost function associated with X'. The two have different cost functions.

Part i:

Rewrite X as $\cap_{r \in R^k_{++}} \{x \in R^k_+ : r \cdot x \geq \mathrm{C}(r)\}$, and you see that X is the intersection of closed and convex sets, hence it itself is closed and convex. And since the r's are strictly positive, it is clearly comprehensive upwards: If $x \in X$ and $x' \geq x$, then $r \cdot x' \geq r \cdot x$ for all r (the r are strictly positive) and $r \cdot x \geq \mathrm{C}(r)$ for all r ($x \in X$), therefore $r \cdot x' \geq \mathrm{C}(r)$ for all r, hence $x' \in X$.

Take any $r^0 \in R^n_{++}$, and let $t = \inf\{r^0 \cdot x : x \in X\}$. Since $x \in X$ only if $r \cdot x \geq \mathrm{C}(r)$

for all r, this must be true in particular for r^0, hence $t \geq \mathrm{C}(r^0)$. We must show that $t = \mathrm{C}(r^0)$ and, to justify the "min," that there is some specific $x^0 \in X$ such that $r^0 \cdot x^0 = \mathrm{C}(r^0)$. Here is where we use the concavity and homogeneity of C: r^0 is in the interior of the domain of C (every strictly positive price vector is), so there is a supergradient of C at r^0; that is, some $x^0 \in R^k$ and scalar β such that $r^0 \cdot x^0 + \beta = \mathrm{C}(r^0)$ and $r \cdot x^0 + \beta \geq \mathrm{C}(r)$ for all $r \in R^k_{++}$. The at-this-point usual argument employing homogeneity is enlisted to show that $\beta = 0$: We know that

$$\alpha r \cdot x^0 + \beta \geq \mathrm{C}(\alpha r) = \alpha \mathrm{C}(r) = \alpha r \cdot x^0 + \alpha\beta \quad \text{or} \quad \beta \geq \alpha\beta,$$

for all $\alpha > 0$, which implies that $\beta = 0$. So we are done once we show that $x^0 \geq 0$. This is where C being nonnegative-valued comes in: Suppose x^0 had a strictly negative component. Let r be the strictly positive price vector with coordinate values 1 everywhere except the one coordinate where x^0 is negative, in which coordinate have r take on the value B, for (very large) positive B. For B large enough, $r \cdot x^0$ will be negative. But $r \cdot x^0 \geq \mathrm{C}(r) \geq 0$, which is a contradiction.

(The problem didn't ask you to prove part j, so I will not include a proof here.)

■ 9.13. I'll deal first with the part about C being continuous, if we assume that $y \Rightarrow V(y)$ is continuous, and then go back to the part where we only assume that the V correspondence is upper semi-continuous.

We are going to apply Berge's Theorem, of course. Continuity of the objective function is obvious, and we've assumed that the constraint correspondence is lower semi-continuous. So the key is to produce the requisite locally bounded and upper semi-continuous sub-correspondence. Here is where the trick is enlisted. Continuity of C and upper semi-continuity of V^* are local properties; if they hold (say) in an open neighborhood of every (y^0, r^0), then they hold "globally." The r argument presents no problem; difficulties that arise come about because of the y argument. So fix any $y^0 \in Y^o$. By the definition of Y^o, some y' exists such that $y' \in Y$ and y' is strictly greater (in every component) than y^0. So we can put an open neighborhood around y^0 such that every y in that neighborhood is strictly less than y'. Now let x^0 be any point in $V(y')$. Because the nesting property holds for the V's, we know that $x^0 \in V(y)$ for all y in the neighborhood of y^0. So for each y in that neighborhood and $r \in R^n_{++}$, consider

$$V'(y, r) = \{x \in V(y') : r \cdot x \leq r \cdot x^0\}.$$

Clearly, since $x^0 \in V(y)$ for all these y, minimizing over $V'(y,r)$ (for a given r) gives the same infimum as does minimizing over $V(y)$, and all the solutions when minimizing over $V(y)$ are contained in $V'(y,r)$. And, following the details of the proof of part g of Proposition 9.24, the correspondence $(y,r) \Rightarrow V'(y,r)$ is upper semi-continuous and locally bounded. So Berge's Theorem can be applied for (y,r) where y is in this neighborhood of y^0 (and $r \in R^n_{++}$), and C is continuous and V^* is upper semi-continuous at y^0 and any r. Since y^0 was an arbitrarily picked point from Y^o, we have the result.

Now suppose that the V correspondence is only upper semi-continuous. Let $\{(y^\ell, r^\ell)\}$ be a sequence from $Y^o \times R^n_{++}$ converging to some $(y^0, r^0) \in Y^o \times R^n_{++}$. Let x^ℓ be any solution to the cost-minimization problem at (y^ℓ, r^ℓ). (We know a solution exists because $V(y^\ell)$ is closed.) The previous paragraph tells us that the x^ℓ live inside a compact set, so by looking along a subsequence if necessary, we can assume that $x^\ell \to x^0$. Upper semi-continuity of the V correspondence tells us that $x^0 \in V(y^0)$, and so $C(y^0, r^0) \leq r^0 \cdot x^0$.

But this tells us that for any sequence $\{(y^\ell, r^\ell)\}$ converging to some (y^0, r^0) in Y$^o \times$ R^n_{++}, lim inf $C(y^\ell, r^\ell) \geq C(y^0, r^0)$. (Start with the sequence, and look along a subsequence along which the limit infimum is attained. Then plug in to the preceding paragraph.) This means that C is lower semi-continous (as a function).