Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 10: The Expenditure-Minimization Problem

Summary of the Chapter

Chapter 9 began by providing results concerning the theory of the competitive and profit-maximizing firm that parallel developments for the theory of the utility-maximizing consumer from Chapters 3 and 4. But after that beginning, Chapter 9 went on to some results that have no parallel in Chapters 3 and 4. The most important of these were: (1) How to reconstruct, to the extent possible, a production-possibility set Z from its profit function. (2) Providing necessary and sufficient conditions for a function π to be a profit function for some production-possibility set Z.

In this chapter and the next, we go back to the theory of the utility-maximizing consumer. A variety of results will be provided, but our first and foremost objective is to replicate those two results from Chapter 9 in the context of the consumer's problem. Unhappily, the consumer's utility-maximization problem is more complex than the firm's profit-maximization problem on two grounds:

1. The firm's profit-maximization problem has the price vector p as parameter. In the consumer's utility-maximization problem, prices enter parametrically, but so does the consumer's income y.

2. In the firm's profit-maximization problem, prices enter the objective function, but the feasible set never changes. In the consumer's utility-maximization problem, prices (and income) shift the feasible set.

In this chapter, we study a problem related to the consumer's utility-maximization problem, the so-called *Expenditure-Minimization Problem*, which deals with the first of these complications while avoiding the second. In the Expenditure-Minimization Problem, the question asked is, Fixing a consumer (identified by her utility function u), for each (attainable) level of utility v and strictly positive price vector p, what is least expensive way at these prices for the consumer to attain utility level v or more. (This problem is sometimes called the *Dual Consumer's Problem*.) The chapter runs the following course:

1. The Expenditure-Minimization Problem, or the EMP, is defined in Section 10.1, and basic analysis of the problem (existence of solutions, convexity of the set of solutions, uniqueness of the solution, homogeneity properties) are provided in Section 10.2.

2. The correspondence that, for each pair (p, v), gives the set of solutions to the EMP at p and v, is named the *Hicksian-demand* correspondence, and the optimized value of the objective function is named the *expenditure function* in Section 10.3. Berge's Theorem is applied. In Section 10.4, basic properties of the expenditure function are derived; most siginficantly, the expenditure function is concave and homogeneous of degree 1 in p for fixed v. Differentiability of the expenditure function in p is shown to be equivalent to uniqueness of solutions to the EMP.

3. A variety of utility functions can give rise to the same expenditure function, and in Section 10.5, we ask and answer the question, When are two utility functions expenditure equivalent, in the sense that they give the same expenditure function? We show in particular that for any continuous utility function u, there is a unique continuous, quasi-concave, and nondecreasing utility function $\hat{u}$ that shares an expenditure function with u. And, in Section 10.6, we discuss how to take an expenditure function and "invert" it to find the unique continuous, quasi-concave, and nondecreasing utility function that generates it.

4. In Section 10.6, we assume that we are given a legitimate expenditure function to invert. In Section 10.7, we find necessary and sufficient conditions for an expenditure function to be "legitimate," in the sense that, when inverted, what results is a continuous, quasi-concave, and nondecreasing utility function. (Local insatiability is included in the list of properties for the utility function.)

5. The chapter concludes in Section 10.8, where we connect the EMP to the Consumer's Problem from Chapters 3 and 4.

Solutions to Starred Problems

■ 10.1. (b) We prove part (b) first (since having the existence of a solution makes the proof for part (a) expositionally neater). The problem is to minimize $p \cdot x$ subject to $u(x) \geq v$, for $u(0) \leq v <$, where = sup $u(x)$. Since $v <$, there exists some x^0 such that $v \leq u(x^0) <$ (the supremum in the definition of is not attained, since u is globally insatiable). Fix such an x^0, and let $X = \{x \in R^k_+ : u(x) \geq v; p \cdot x \leq p \cdot x^0\} = \{x \in R^k_+ : u(x) \geq$

$v\} \cap \{x \in R^k_+ : p \cdot x \leq p \cdot x^0\}$. Both sets in the intersection are closed, the first because u is a continuous function and the second because $p \cdot x$ is continuous in x. And the second set is bounded; p is strictly positive, so the arguments we gave back in Chapter 3 for the boundedness of budget sets work. Therefore, X is the intersection of two closed sets, one of which is bounded, and X is compact. Hence, the problem of minimizing $p \cdot x$ over X has a solution (minimizing a continuous function $x \to p \cdot x$ over a compact set). But any solution to this problem is a solution to the EMP for p and v: If x is feasible for the EMP, then $u(x) \geq v$, and if x is a solution, then $p \cdot x \leq p \cdot x^0$, since x^0 is feasible. Therefore, if x solves the EMP, then $x \in X$. On the other hand, every point in X is feasible for the EMP; indeed, X is a subset of the set of feasible points for the EMP, so minimizing over X cannot improve matters. That is, solutions to the EMP are solutions to minimizing $p \cdot x$ over X (and vice versa), and there are solutions to the problem of minimizing $p \cdot x$ over X, therefore there are solutions to the EMP.

(a) Suppose x solves the EMP for prices p and target utility v but not for λp and v, for some $\lambda > 0$. Let x' be a solution to the EMP for λp and v (a solution exists by part b). Then $\lambda p \cdot x' < \lambda p \cdot x$, which (since $\lambda > 0$) implies that $p \cdot x' < p \cdot x$. But if x' is a solution to the EMP for λp and v, then $u(x') \geq v$, so x' is feasible for the EMP at p and v, in which cas $p \cdot x' < p \cdot x$ contradicts the original hypothesis that x solves the EMP for p and v. This contradiction proves that x must solve the EMP for λp and v (if it solves the EMP for p and v).

(c) Suppose x and x' solve the EMP at p and v. Then x and x' must both be feasible for this problem, which means that $u(x) \geq v$ and $u(x') \geq v$. For any $\alpha \in [0, 1]$, $u(\alpha x + (1-\alpha)x') \geq \min\{u(x), u(x')\} \geq v$, by quasi-concavity of u, and therefore $\alpha x + (1-\alpha)x'$ is feasible for the EMP at p and v. But $p \cdot (\alpha x + (1-\alpha)x') = \alpha p \cdot x + (1-\alpha) p \cdot x' = p \cdot x = p \cdot x'$, since $p \cdot x = p \cdot x'$ (they are both solutions), which implies that $p \cdot (\alpha x + (1-\alpha)x')$ has the same (minimal) level of expenditure; it is also a solution. The set of solutions is convex.

And if u is strictly quasi-concave: If x and x' are distinct solutions of the EMP at p and v, one of them is not 0, and since p is strictly positive, $p \cdot x = p \cdot x' > 0$; that is, both are not 0. Let $x'' = 0.5x + 0.5x'$; by strict quasi-concavity, $u(x'') > \min\{u(x), u(x')\} = v$. But then by continuity, for $\beta < 1$ but sufficiently close to 1, $u(\beta x'') > v$, and $\beta x''$ is feasible. Now $p \cdot (\beta x'') = \beta p \cdot x'' < p \cdot x = p \cdot x'' = p \cdot x'$ (since $p \cdot x = p \cdot x' = p \cdot x'' > 0$), which (since $\beta x''$ is feasible) contradicts the alleged optimality of x and x'. There cannot be two distinct solutions if u is strictly quasi-concave.

(d) If x solves the EMP at p and v, then x must be feasible; that is, $u(x) \geq v$. So if $u(x) \neq v$, it must be that $u(x) > v$. Since $v \geq u(0)$, $x \neq 0$, and $p \cdot x > 0$. But then by continuity of u, for β strictly less than 1 but close to 1, $u(\beta x) > v$; that is, βx is feasible. And $p \cdot (\beta x) = \beta p \cdot x < p \cdot x$, which means that x isn't optimal for p and v, a contradiction. It must be that $u(x) = v$.

▪ 10.3. The first step is to show that for all $p \in R^k_{++}$ and $v \in R_+$, then there exists some $x \in R^k_+$ that attains the infimum in the definition of e. Since, by assumption, sup $u(x) = \infty$, there exists some $x^0 \in R^k_+$ such that $u(x^0) \geq v$. By the same argument as was given in the proof of Proposition 10.2(b) (see the solution just given), if we let $X = \{x \in R^k_+ : u(x) \geq v \text{ and } p \cdot x \leq p \cdot x^0\}$, then the set of solutions (if any) to the problem of finding the infimum of $p \cdot x$ over all x such that $u(x) \geq v$ is the same as the set of solutions to the problem of finding the infimum over $x \in X$. And $X = \{x \in R^k_+ : u(x) \geq v\} \cap \{x \in R^k_+ : p \cdot x \leq p \cdot x^0\}$. The second set in the intersection is compact (again see the solution given just previously), while the first set is closed, since u is upper semi-continuous. Therefore we are minimizing a continuous function over a compact set, and a solution exists; the infimum is attained.

Moveover, $p \to e(p, v)$ is concave in p for each fixed v. The argument from the text (Proposition 10.4(d)) works like a charm: Fix p, p' and $p'' = \alpha p + (1 - \alpha)p'$, for $\alpha \in [0, 1]$. Let x'' solve the minimization problem for p'' (a solution exists per the previous paragraph). Then $u(x'') \geq v$, so x'' is feasible for (p, v) and (p', v), and $e(p, v) \leq p \cdot x''$ and $e(p', v) \leq p' \cdot x''$. But then

$$\alpha e(p, v) + (1 - \alpha)e(p', v) \leq \alpha(p \cdot x'') + (1 - \alpha)p' \cdot x'' = (\alpha p + (1 - \alpha)p') \cdot x'' = e(\alpha p + (1 - \alpha)p'', v),$$

showing that e is concave in p. And, being concave in p (since p is drawn from an open domain), e is continuous in p.

It remains to show that e is lower semi-continuous in (p, v). Let $\{(p^n, v^n)\}$ be a sequence of price–target utility pairs with limit (p, v). Let x^n be a solution to the minimization problem at (p^n, v^n); that is, $u(x^n) \geq v^n$ and $p^n \cdot x^n = e(p^n, v^n)$. To show lower semi-continuity, we have to show that

$$e(p, v) \leq \liminf e(p^n, v^n).$$

Look along a subsequence, along which the limit infimum is attained. (I'll assume that the original sequence is that subsequence.) If the limit infimum is $+\infty$, then there is nothing more to show. So suppose the limit infimum is finite. This means that the set $\{e(p^n, v^n); n = 1, \ldots\}$ is bounded uniformly by some M. Moreover, since $p^n \to p$ and p is strictly positive, there is a lower bound m on all the components p^n_i for $i = 1, \ldots, k$ and $n = 1, \ldots$. And since $M \geq e(p^n, v^n) = p^n \cdot x^n$, this means there is an upper bound M/m on the components x^n_i for $i = 1, \ldots, k$, $n = 1, \ldots$. But then the vectors x^n live in a compact set, and by looking along a (further) subsequence, we can assume that x^n converges to some x. Since u is upper semi-continuous, $u(x) \geq \limsup u(x^n) \geq \lim v^n = v$, and therefore x is feasible at (p, v), so $e(p, v) \leq p \cdot x$. But by continuity of the dot product, $p \cdot x = \lim p^n \cdot x^n = \lim e(p^n, v^n)$, showing that e is lower semi-continuous.

■ 10.6. The problem asks how much of Proposition 10.18 survives if u is not necessarily locally insatiable and (only) upper semi-continuous. So to begin, we note that as long as u is upper semi-continuous, solutions to both the CP and the EMP are guaranteed to exist (for strictly positive prices, of course). The former is true because in the CP, we are maximizing an upper semi-continuous function on a compact set, which is enough for existence of a solution. As for the EMP, see the first paragraph in the solution to Problem 10.3 just given.

So fix p and y. Suppose $x \in (p, \nu(p, y))$. Let x' be any solution of the CP at p and y; so (of course) $p \cdot x' \leq y$ and $u(x') = \nu(p, y)$. The second of these conclusions tells us that x' is feasible for the EMP at p and $\nu(p, y)$, and (hence) x, being a solution of the EMP, must be no more expensive, or $p \cdot x \leq p \cdot x' \leq y$, which means that x is feasible for the CP. But $u(x) \geq \nu(p, y)$, which tells us that x must be a solution to the CP at p and y, or $(p, \nu(p, y)) \subseteq (p, y)$. Moreover, $e(p, \nu(p, y)) = p \cdot x \leq y$.

The following example shows that this set inclusion may be strict and that $e(p, \nu(p, y)) < y$ is possible. Take $k = 1$ and let

$$u(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 5, \\ 10 - x, & \text{for } 5 < x \leq 6, \text{ and} \\ x - 2, & \text{for } x > 6. \end{cases}$$

(Draw the function u if you don't "see" it.) The key is that this function u has a local max at $x = 5$. Now for $p = 1$ and $y = 7$, there are two solutions to the CP, namely $x = 5$ and $x = 7$, giving utility level 5. But the cheapest way to achieve utility level 5 is with $x = 5$ only. This gives both things that we wanted to show.

Note that u in this example is continuous; it is the lack of local insatiability that causes the "one-way" implications. In particular, suppose that u is only upper semi-continuous but is also locally insatiable. And suppose that $x \in (p, y)$ but is not in $(p, \nu(p, y))$. This implies that there exists $x' \in (p, \nu(p, y))$ which is strictly cheaper than x at prices p; that is $p \cdot x' < p \cdot x$. By local insatiability, there exists x'' that is strictly preferred to x' but still is cheaper than x at prices p; that is, $u(x'') > u(x') \geq \nu(p, y)$ and $p \cdot x'' \leq p \cdot x \leq y$, which contradicts the supposed optimality (in the CP) of x. So if u is locally insatiable but (only) upper semi-continuous, then $(p, \nu(p, y)) = (p, y)$, and $e(p, \nu(p, y)) = y$.

Now fix p and v. Suppose $x \in (p, e(p, v))$. Then $p \cdot x \leq e(p, v)$. Suppose $x' \in (p, v)$. We know that $p \cdot x' = e(p, v)$, so x' is feasible for the CP at p and $e(p, v)$. Since x is optimal for the CP at those parameters, $u(x) \geq u(x') \geq v$, and x is feasbile for the EMP at p and v. But since x is feasible for the CP at p and $e(p, v)$, $p \cdot x \leq e(p, v)$, and (therefore) $x \in (p, v)$. This implies that $(p, e(p, v)) \subseteq (p, v)$. Moreover, $\nu(p, e(p, v)) = u(x) \geq v$.

If we know that u is continuous, the set inclusion and inequalities just given are equalities: First, if u is continuous and $x \in (p, v)$, we know that $u(x) = v$. But $u(x)$ here is $\nu(p, e(p, v))$. And, suppose $x' \in (p, v)$. Since $p \cdot x' = e(p, v)$ by definition, x' is feasible for the CP at p and $e(p, v)$. Let x be any arbitrary element of $(p, e(p, v))$. We proved that $x \in H(p, v)$ and so, since u is continuous, $u(x) = v$. Similarly, we know that $u(x') = v$. But then x' is feasible for the CP and provides as much utility as a solution to the CP, so $x' \in (p, e(p, v))$.

To finish off, we need an example where u is upper semi-continuous (only) and locally insatiable, and for which the set inclusion $(p, e(p, v)) \subseteq (p, v)$ and inequality $\nu(p, e(p, v)) \geq v$ are both strict. It takes two dimensions to do this, so the example is a bit complex: Let

$$u((x_1, x_2)) = \begin{cases} x_1 + x_2 + 1, & \text{for } x_1 + x_2 \geq 1 \text{ and } x_2 \geq x_1 \text{, and} \\ x_1 + x_2, & \text{otherwise.} \end{cases}$$

Set $p = (1, 2)$ and $v = 1.5$. To minimize expenditure and achieve this level of utility, there are two solutions: $(0.5, 0.5)$ and $(1.5, 0)$, each costing 1.5. But at these prices and a budget of 1.5, only $(0.5, 0.5)$ is utility maximizing, giving an indirect utility level of 2.

■ 10.8. If x^0 is a solution to the MEMP(p, x^0), then it is also a solution to the MCP(p, x^0), but not vice versa:

Suppose x^0 minimizes $p \cdot x$ subject to $u(x) \geq u(x^0)$. If $x^0 = 0$, then $p \cdot x^0 = 0$, and $x^0 = 0$ is the only feasible consumption bundle in the MCP(p, x^0), so is optimal. So we can assume that $x^0 \neq 0$ and $u(x^0) > u(0)$ (since if $x^0 \neq 0$ but $u(x^0) \leq u(0)$, then x^0 cannot be a solution to the MEMP(p, x^0), as 0 costs less at strictly positive prices and satisfies the utility constraint).

Now suppose in addition that x^0 does not maximize $u(x)$ subject to $p \cdot x \leq p \cdot x^0$. Then there exists some x^* such that $p \cdot x^* \leq p \cdot x^0$ but $u(x^*) > u(x^0) \geq u(0)$. Since $u(x^*) > u(0)$, $x^* \neq 0$. But then for β strictly less than one but close to one, continuity of u ensures that $u(\beta x^*) > u(x^0)$, and βx^* is feasible for the MEMP(p, x^0). This gives us a contradiction to the original assumption that x^0 is a solution to the MEMP(p, x^0), since βx^* costs strictly less than x^* (remember that $x^* \neq 0$), which in turn costs no more than x^0 at the prices p.

Now take the counterexample from the solution to Problem 10.7, namely $k = 1$ and let

$$u(x) = \begin{cases} x, & \text{for } 0 \leq x \leq 5, \\ 10 - x, & \text{for } 5 < x \leq 6, \text{ and} \\ x - 2, & \text{for } x > 6. \end{cases}$$

I assert that, for $p = 1$, 7 is a solution to the MCP$(p, 7)$, since with a budget of 7 and $p = 1$, the best bundles are 5 and 7, both giving utility 5. But with a target utility $u(7) = 5$, the cheapest bundle is 5; 7 is not a solution of MEMP$(p, 7)$.

Clearly, we need local insatiability of u. Suppose u is locally insatiable and x^0 is a solution of the MCP(p, x^0). Then we know that x^0 maximizes $u(x)$ on the set of x such that $p \cdot x \leq p \cdot x^0$. Suppose, however, x^0 does not minimize $p \cdot x$ on the set of x such that $u(x) \geq u(x^0)$. Then there is some x^* with $p \cdot x^* < p \cdot x^0$ and $u(x^*) \geq u(x^0)$. By local insatiability, we can find a close neighbor of x^*, call it x', close enough so that $p \cdot x' < p \cdot x^0$ and $u(x') > u(x^*) \geq u(x^0)$. But this contradicts the supposed optimality of x^0 in the MCP(p, x^0).