Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 13: Aggregating Firms and Consumers

Summary of the Chapter

This chapter is primarily concerned with the following two questions:

- Suppose we have a finite number F of profit-maximizing firms and, more specifically, their optimal netput correspondences $p \Rightarrow Z^{f*}(p)$ and profit functions $p \to \pi^f(p)$, telling us what are the (possibly multiple) optimal production plans for the firms and their levels of profit as a function of the price vector p. We form the *aggregate optimal netput correspondence* $p \Rightarrow Z^*(p)$, defined by

  $$Z^*(p) := \{z \in R^k : z = z^1 + \ldots + z^F \text{ for some selection of } z^f \in Z^{f*}(p), f = 1, \ldots, F\},$$

  and the aggregate profit function

  $$\pi(p) := \sum_{f=1}^{F} \pi^f(p).$$

  Chapter 9 tells us a lot about $p \Rightarrow Z^{f*}(p)$ and $p \to \pi^f(p)$, if these come from a profit-maximizing firm. What can we say in this vein about the properties of the aggregate optimal netput correspondence and aggregate profit function?

- And suppose we have a finite number H of utility-maximizing consumers. Let $\mathbf{D}^h(p, y^h)$ be the Marshallian demand of consumer h, as a function of prices p and income y^h. Form the aggregate demand for these H consumers by summing up across consumers their Marshallian demands (sum a selection, one from each $\mathbf{D}^h(p, y^h)$, if $\mathbf{D}^h(p, y^h)$ is not singleton). Chapter 11 tells us a lot about the structure of individual consumer demand. What can we say in this vein about aggregate demand?

The answers obtained are

1. For firms, aggregation works like a charm. Aggregate optimal-netput from a (finite) number of competitive firms has precisely the properties of optimal-netput from a single competitive firm, where the "aggregate firm" is constructed as a simple merger of the production possibilities of the firms being aggregated.
2. But for consumers, things do not work very well. In some extraordinarily special cases, one can aggregate. But in general, the *Sonnenschein–Mantel–Debreu Theorem* tells us (more or less) that any continuous function from prices to excess demands that is continuous, homogeneous of degree zero, and that satisfies Walras' Law is the aggregate excess demand for a collection of consumers with continuous, nondecreasing, and convex preferences.
3. Finally, and in a different vein, aggregation can "smooth" out what is being aggregated. We illustrate with a result due to Shapley, Folkman, and Starr, which shows (for instance) that per capita aggregate demand correspondences become "asymptotically" convex-valued as the number of consumers increases, if the number of commodities stays fixed. (This result requires that some assumptions hold, of course.)

Solutions to Starred Problems

▪ 13.1. If preferences are only identical and homothetic, the proposition as stated won't work. Suppose, for instance, $k = 2$, there are two consumers, and their preferences are given by $u((x_1, x_2)) = \max\{x_1, x_2\}$. Then for the price vector $p = (1, 1)$, $(p, y) = \{(y, 0), (0, y)\}$. Suppose1 we split y into y^1 and y^2 such that $y^1 + y^2 = y$; we get

$$(p, y^1) + (p, y^2) = \{(y, 0), (y^1, y^2), (y^2, y^1), (0, y)\}.$$

The Minkowski sum of the (two) individual demands will depend on how income is split up. One can show that the Minkowski sum of the individual demands is always a subset of Marshallian demand for the (homothetic) preferences that are given by the convexified preferences derived from the original preferences (see Proposition 10.13, although to plug directly into this, we should assume that the original preferences are nondecreasing, as well). But with identical and homothetic but nonconvex preferences, this is about the best you can do.

▪ 13.4. (a) The scalars β^h just scale the objective function and can be ignored. If $y = 0$, then the only feasible point is 0 (the origin in R^H), so the result is true. Now suppose that $y > 0$. We know that, at any solution, $y^h > 0$ for all h, since otherwise you get 0

for the value of the objective function, and strictly positive values of the objective function are clearly feasible. And, having recognized this, we know that any solution will be interior. Now you can proceed to look at the first-order conditions for a maximum, although to make life even easier, subject the objective function to the monotonic transformation of taking its natural log. The problem (ignoring the β^h, remember) becomes

$$\max \sum_h \alpha^h \ln(y^h), \quad \text{subject to } \sum_h y^h \le y.$$

This is a strictly concave objective function (so was the other, but it is easier to see it in this form), so there is a unique solution, and first-order, complementary-slackness conditions are $\alpha^h/y^h = \lambda$, where λ is the mulitplier on the constraint. This is $y^h = \alpha^h/\lambda$, and to satisfy the constraint, this is $\lambda = 1/y$ so that $y^h = \alpha^h y$.

(b) Since u^h is, by assumption, homogeneous of degree 1, we know that the solution for y is just the solution for $y = 1$ scaled up by y; that is, the solution is $y\hat{x}^h$ and the value of the objective function at the solution is $\beta^h y$. (If there are multiple solutions, they all scale, but we'll fix on one, namely $\hat{x}^h$.) It is worthwhile observing at this point that $\beta^h > 0$ by virtue of the assumption that there is some x such that $u^h(x) > u^h(0) = 0$, which also implies that the consumer spends all of her income at the solution.

(c) For each h, we know that $u^h(y\hat{x}^h) \ge u^h(x)$ for any x that satisfies $p \cdot x \le y$. So if we set $\check{y}^h = p \cdot \check{x}^h$, we know that $u^h(\check{y}^h\hat{x}^h) \ge u^h(\check{x}^h)$ for each h. This implies that

$$U(x^0) = \prod_h (u^h(\check{x}^h))^{\alpha^h} \le \prod_h (u^h(\check{y}^h\hat{x}^h))^{\alpha^h} = \prod_h (\check{y}^h\beta^h)^{\alpha^h}.$$

(d) But since $p \cdot \hat{x}^h = 1$, $p \cdot \left[\sum_h \alpha^h y\hat{x}^h\right] = \sum_h \alpha^h y = y$. So if we let $\hat{x} = \sum_h \alpha^h y \hat{x}^h$, we know that $p \cdot \hat{x} \le y$, so $U(x^0) \ge U(\hat{x})$. Moreover,

$$\begin{aligned} U(x^0) \ge U(\hat{x}) &= \max\left\{ \prod_h (u^h(\tilde{x}^h))^{\alpha^h} : \sum_h \tilde{x}^h \le \hat{x} \right\} \\ &\ge \prod_h (u^h(\alpha^h y\hat{x}^h))^{\alpha^h} = \prod_h (\alpha^h y \beta^h)^{\alpha^h}. \end{aligned}$$

(e) But by part a, since both $\sum_h \alpha^h y = y$ and $\sum_h \check{y}^h \le y$, we know that

$$\prod_h (\alpha^h y\beta^h)^{\alpha^h} \ge \prod_h (\check{y}^h\beta^h)^{\alpha^h},$$

and moreover that the two will be equal only if $\check{y}^h = \alpha^h y$ for each h. But the string of inequalities we've produced say they must be equal. Therefore, $\check{y}^h = \alpha^h y$ for each y,

$\hat{x}$, which is the sum of individual demands, must be a solution the consumer's problem for the aggregate demand function, and each individual $\check{x}^h$ must solve each individual consumer's demand problem at prices p and income $\check{y}^h = \alpha^h y$, for if any fell short of the utility generated by $\alpha^h y \hat{x}^h$, then the string of equalities would fail. ■

■ 13.5. Here's an analogous result to Proposition 13.4, but stated in terms of excess demand:

Consider the problem

$$\text{Maximize } u^h(\zeta + e^h), \text{ subject to } p \cdot \zeta \leq 0,\ \zeta \in R^k,\ \zeta \geq -e^h$$

where u^h is a continuous utility function defined on R^k_+, $p \in R^k_{++}$, $e^h \in R^k_+$. Let $^h(p, e^h)$ denote the set of solutions for this problem, and let $\eta^h(p, e^h)$ denote sup $u^h(\zeta + e^h)$ subject to the same constraints. Then:

a. *A solution exists for each p and e^h; that is, $^h(p, e^h)$ is nonempty. Therefore the supremum that defines $\eta^h(p, e^h)$ is a maximum and $\eta^h(p, e^h)$ is finite for each p and e^h.*

b. *If u^h is quasi-concave, the $^h(p, e^h)$ is convex. If u^h is strictly quasi-concave, then $^h(p, e^h)$ is singleton.*

c. *The correspondence $(p, e^h) \Rightarrow^h (p, e^h)$ is upper semi-continuous and locally bounded, and the function $(p, e^h) \to \eta^h(p, e^h)$ is continuous.*

d. *$^h(p, e^h) =^h (\lambda p, e^h)$ for all $\lambda > 0$, and $p \to \eta(p, e^h)$ is homogeneous of degree 0.*

e. *If u^h is locally insatiable, then $p \cdot \zeta = 0$ for all $\zeta \in^h (p, e^h)$.*

It remains in this problem to prove Proposition 13.6:

Suppose $\zeta \in (p)$. Then $\zeta = \zeta^1 + \ldots + \zeta^H$ where each $\zeta^h \in^h (p)$. (The endowments are fixed throughout at e^h for consumer h.) For any $\lambda > 0$, $\zeta^h \in^h (\lambda p)$ (see d immediately above), so $\lambda\zeta = \lambda\zeta^1 + \ldots + \lambda\zeta^H \in (\lambda p)$. Reversing the argument for $\zeta \in (\lambda p)$ where we scale everything with $1/\lambda$ shows that $\zeta \in (p)$, and so $(p) = (\lambda p)$ for all $\lambda > 0$.

Suppose $\zeta \in (p)$. Then $\zeta = \zeta^1 + \ldots + \zeta^H$, where each $\zeta^h \in^h (p)$. But then $p \cdot \zeta^h = 0$ for each h (see e above, and recall that the proposition assumes that each consumer is locally insatiable), hence $0 = p \cdot \zeta^1 + \ldots p \cdot \zeta^H = p \cdot (\zeta^1 + \ldots + \zeta^H) = p \cdot \zeta$. Walras' Law holds.

Suppose each consumer has convex preferences. Fix $\zeta, \zeta' \in (p)$. Then $\zeta = \zeta^1 + \ldots + \zeta^H$ and $\zeta' = \zeta'^1 + \ldots + \zeta'^H$ where each ζ^h and $\zeta'^h \in^h (p)$. But since each $^h(p)$ is convex (see b above), for any $\lambda \in [0, 1]$, $\lambda\zeta^h + (1-\lambda)\zeta'^h \in^h (p)$, hence $\lambda\zeta + (1-\lambda)\zeta' = \sum_h \big(\lambda\zeta^h + (1-\lambda)\zeta'^h\big) \in (p)$. That is, (p) is convex.

Finally, to show that $p \Rightarrow (p)$ is upper semi-continuous: Suppose that $p^n \to p^0$ and $\zeta^n \to \zeta^0$ where each $\zeta^n \in (p^n)$. Write $\zeta^n = \zeta^{n1} + \ldots + \zeta^{nH}$, where $\zeta^{nh} \in^h (p^n)$. I assert that $\{\zeta^{nh}\}_{n=1,\ldots}$ lies inside a bounded set for each h. This follows from the fact that each $p \Rightarrow^h (p)$ is locally bounded, but let me give details, since they are simple: Since $p^n \to p$ and p is strictly positive, if we let $\epsilon = \min_i \big(p_i / \sum_{i'} p_{i'}\big)$, we know that for all sufficiently large n, $p^n_i / \left(\sum_{i'} p^n_{i'} \right) \geq \epsilon/2$. We know that $\zeta^{nh} \geq -e^h$, so if we let $M = \max_{i=1,\ldots,k; h=1,\ldots,H} e^h_i$, we know that $\zeta^{nh}_i \geq -M$ for all n, h, and i. But since $p^n \cdot \zeta^{nh} = 0$, this implies that for all n large enough so that $p^n_i / \left(\sum_{i'} p^n_{i'} \right) \geq \epsilon/2$, $\zeta^{nh}_i \leq M/(\epsilon/2) = 2M/\epsilon$. (If this were not so for some n, h, and i, then the contribution $p^n_i \zeta^{nh}_i$ to the dot product would be too large (and positive) to be counteracted by the contribution of all the other terms; the dot product would have to be strictly positive.) We already know that $-M$ is a (uniform) lower bound on the components of any $\zeta^h(p)$; this gives us a local upper bound.

Of course, once we know that each individual excess demand correspondence is locally bounded, we know that aggregate excess demand is also locally bounded.

And once we know that each $\{\zeta^h_n\}_{n=1,\ldots}$ lies within a bounded set, we can extract a subsequence along which $\{(\zeta^1_n, \ldots, \zeta^H_n)\}$ converges to some $(\zeta^1_0, \ldots, \zeta^H_0)$, and by the upper semi-continuity of $p \rightrightarrows^h (p)$ for each h, we know that each ζ^h_0 lies in ${}^h(p_0)$. But $\zeta_0 = \lim_n(\zeta^1_n + \ldots + \zeta^H_n)$ and, along the subsequence, this is equals $\lim_n(\zeta^1_n + \ldots + \zeta^H_n) = \zeta^1_0 + \ldots + \zeta^H_0$. Hence $\zeta_0 \in (p_0)$.

■ 13.7. For the set $X = \{1, 2, 3, 4\}$ in R^1, a point $x \in (X)$ is of greatest distance from points in X that (minimally) have x as a convex combination are points that are very close one of the points in X and so are very far from the "neighboring" point in X. That is, the worst case are points as $2 + \epsilon$ for very small $\epsilon > 0$. If you write this as a convex combination of points in the set, the most economical (in terms of distance from the further point) is to write it as a convex combination of 2 and 3, and the distance from 3 is $1 - \epsilon$. Hence the inner radius of this set is 1.

For the set $X = \{(z_1.z_2) \in R^2 : z_1 = 1, 2, 3, \text{ or } 4 \text{ and } z_2 = 1, 2, 3, \text{ or } 4\}$, the "worst case" $x \in (X)$ is where you are very close to but not on a lattice point in either direction. For instance, consider the point $(2 + \epsilon, 2 + \delta)$, $\delta \neq \epsilon$, $\delta, \epsilon > 0$. To make that a convex combination, we'll require $(2, 2)$ and $(3, 3)$ and either $(2, 3)$ or $(3, 2)$, depending on whether δ or ϵ is larger. (If $\delta = \epsilon$, we'd only need $(2, 2)$ and $(3, 3)$, although this won't reduce the calculation of "furthest needed point.") The furthest point is $(3, 3)$ (if both δ and ϵ are small), with a distance that approaches $\sqrt{2}$ as δ and ϵ go to zero. So $\sqrt{2}$ is the inner radius of the set.