Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 14: General Equilibrium

Summary of the Chapter

This chapter and the two to follow conclude this volume by looking at General Equilibrium Theory. We formulate economies populated by (finitely) many consumers and, perhaps, firms, and look at how the various economic actors interact via prices and markets, under the assumption that the consumers and firms are all price takers. We look for and at *Walrasian equilibria*, consisting of a price vector, consumption allocations for all the consumers, and production plans for all the firms, where each consumer's consumption allocation solves her utility-maximization problem at the given prices, each firm's production plan solves its profit-maximization problem at those prices, and markets clear, meaning that the demand for each and every good is no greater than its supply.

In this chapter, we are concerned with formulation, basic properties of Walrasian equilibria, the existence of at least one equilibrium, and (somewhat briefly) the mathematical nature of the set of equilibria for a given economy. Next chapter takes up issues of the efficiency of Walrasian equilibria, and Chapter 16 concludes by adapting the formalisms of this chapter to situations involving multiple periods and uncertainty.

More specifically for this chapter:

1. Section 14.1 provides basic definitions: the formal structure of an *economy*, both with firms and without (so-called *pure-trade* economies); and a precise definition of *Walrasian equilibrium*, the object of our attention for most of the remainder of the book.

2. In Section 14.2, basic properties of Walrasian equilibria are proved. In particular, we give conditions under which *Walras' Law* holds (the market value of what consumers consume equals the market value of all their resources) and under which equilibrium prices must be nonnegative and, then, strictly positive.

3. The *Edgeworth Box* is the topic of Section 14.3. This is a graphical depiction of a two-consumer, two-commodity pure-exchange economy, very useful for developing both intuition and illustrative examples.

4. Section 14.4 is the heart of Chapter 14, concerned with the question, Under what conditions on a given economy can we be sure that at least one Walrasian equilibrium exists. The approaches to existence taken in this chapter employ fixed-point theorems and, in particular, Kakutani's Fixed-Point Theorem, concerning which you should consult Appendix 8. Two general approaches are taken:

   a. First we take the approach of one of the two seminal papers in the subject, Arrow and Debreu (1954). In this approach, we define a so-called generalized game and prove an existence result for Nash equilibria of these games. This is then applied to the question of existence, where the economy (consisting of consumers and firms) is explicitly and formally specified.

   b. And then we give two existence results more in the spirit of the second seminal paper on the subject, McKenzie (1954). In this approach, consumers and firms are implicit; we begin formally with an aggregate excess-demand correspondence. We state and prove what is known as the Debreu-Gale-Kuhn-Nikaido Lemma, for aggregate excess-demand correspondences that are defined for all nonnegative price vectors. And we state and prove an existence result for aggregate excess-demand defined (only) for strictly positive prices due to Hildenbrand (1974).

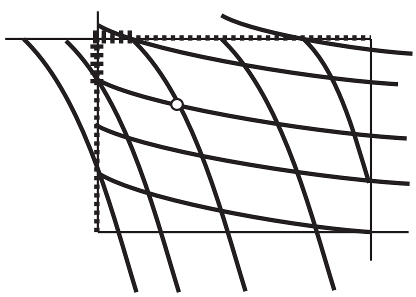
5. Having provided results that guarantee the existence of at least one equilibrium, we take the opposite tack in Section 14.5: What can be said to ensure that the number of equilibria isn't "too large?" Very little is proved in this section; this is more of a discussion than an explication of results. We argue that the set of equilibrium prices (normalized to lie in the unit simplex) for a given economy must be closed, but we then cite Mas-Collel's (1977) Theorem: For *any* closed set of prices in the unit simplex that does not intersect the boundary, there is a pure-exchange economy whose set of Walrasian equilibrium prices is that set.

6. Finally, we state (and leave for you to prove) a result along the lines of: the Walrasian equilibrium correspondence is upper semi-continuous in the parameters of the economy. *But this proposition (Proposition 14.14 on page 354) is not true as stated, on two grounds, one easily fixed, the other not. Immediately following this introductory discussion, I address this error in the text.*

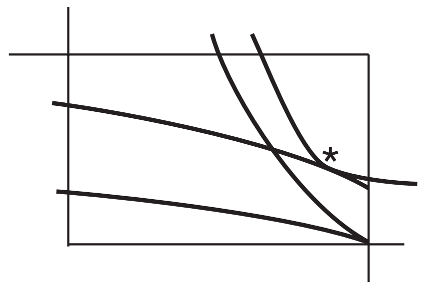
The level of abstraction in this chapter is very high, making it somewhat easy for students to lose sight of what all the symbols mean. For this reason, I strongly recommend that, after you absorb sections 14.1 and 14.2, you take the time to compute the Walrasian equilibria of a few parametric examples of economies; solve Problems 14.2 through 14.5.

Solutions to Starred Problems

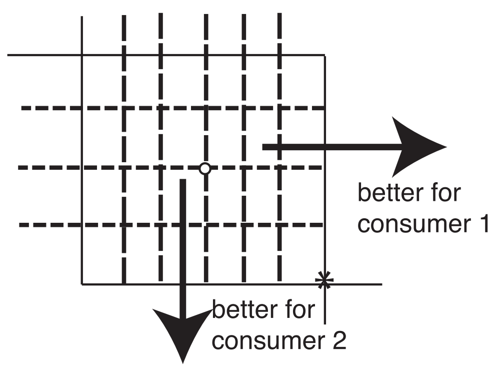
▪ 14.1. Figure G14.1, panel a, shows indifference curves that give a Pareto-efficient set lying entirely along the west and north boundaries of the Edgeworth box. Both parties value both goods, and preferences are strictly convex, but consumer 1 is willing to give up a lot of good 1 for a little good 2, while consumer 2 has the opposite tradeoffs.



(a) Pareto-efficient divisions all lie along the western and northern boundaries of the Edgeworth box (dashed lines) and the core consists of pieces of these two boundaries in the north-west (heavy dashed lines)



(a') The south-east corner is Pareto efficient, but there are also Pareto-efficient divisions that give strictly positive amounts to each consumer, such as the point marked with the asterisk



(b) If consumer 1 only values good 1, and consumer 2 only values good 2, then it is Pareto efficient to give all of good 1 to consumer 1 and all of good 2 to consumer 2. The point marked with the asterisk is the only Pareto-efficient allocation, so it is the only core allocation

Figure G14.1. Solutions to Problem 14.1

Panel a′ provides the other example of part (a) of the problem: The south-east corner is Pareto efficient, because at that point, consumer 1′s rate of trading off good 1 for good 2 (along the indifference curve) is much smaller than consumer 2′s. But, at least as drawn, consumer 1′s indifference curve at a higher level of utility changes shape radically, enough so that now an efficient point gives strictly positive amounts of each good to each consumer.

And panel b shows what things look like if consumer 1 only values good 1 and consumer 2 only values good 2. The only Pareo-efficient point is the south-east corner, and since any point in the core must be efficient, this point, marked with an asterisk in the picture, is the only core allocation. And, of course, the unique Walrasian equilibrium has $p_1 = p_2$, so that each consumer has wealth $20p_1 = 20p_2$, with consumer 1 demanding (and getting) $(20, 0)$ and consumer 2, $(0, 20)$.

■ 14.2. (a) Normalize prices to sum to 1, and let p be the price of the first good. Note that this means that Alice has initial wealth 10, while Bob's wealth is $5p + 10(1 - p) = 10 - 5p$. We solve the consumers' problems in the usual fashion, leading to the following demand functions:

$$x_1^A = \frac{4}{p}, \quad x_2^A = \frac{6}{1-p}, \quad x_1^B = \frac{10-5p}{2p}, \quad \text{and } x_2^B = \frac{10-5p}{2(1-p)}.$$

If you equate supply to total demand and solve for the equilibrium prices, you will find

$$p = \frac{18}{35}, \quad x_1^A = \frac{70}{9}, \quad x_2^A = \frac{210}{17}, \quad x_1^B = \frac{65}{9}, \quad \text{and } x_2^B = \frac{130}{17}.$$

These are the only solutions to supply equals demand: This economy has a single Walrasian equilibrium.

■ 14.3. I assert that at any Walrasian equilibrium of this economy, the price of all three goods must be strictly positive. First, Alice and Bob have strictly increasing utility functions in both consumption goods, so the first part of Proposition 14.5 applies, giranteeing that they have strictly positive prices. And then the second part of the proposition applies to the input good.

That being so, normalize prices so the price of good 3 is 1.

Alice and Bob will have strictly positive wealth. The value of their endowments will be 5, and firms can avoid taking losses (since $(0, 0, 0)$ is feasible for them), so each consumer's wealth will be at least 5. In fact, their wealth levels will be 5, because both firms have constant-returns-to-scale technologies: If prices allowed either firm to make positive profit, it would seek to make infinite profit, which is not possible in a Walrasian equilibrium.

Indeed, with p_3 normalized to be 1, we know that $p_1 \leq 1/3$ and $p_2 \leq 1/4$ are required (to avoid infinite profit opportunities). More than that, if $p_1 < 1/3$, then firm 1 will definitely choose not to produce good 1. That can't be part of an equilibrium: At any price for good 1, Alice and Bob (with wealth 5 each) both demand some good 1, and the only source for good 1 in this economy is production by firm 1. So we know that in any Walrasian equilibrium (with p_3 normalized to be 1), p_1 must be $1/3$. Similarly, p_2 must be $1/4$, to induce firm 2 to produce some of good 2, so that markets clear.

And now that we know the equilibrium price vector (namely, $(1/3, 1/4, 1)$), it is a simple matter to work out demands by Alice and Bob, therefore the amounts that the firms must produce of their respective goods, therefore their demands for good 3 input. The solution on the consumption side is $x_1^A = 6, x_2^A = 12, x_1^B = 7.5$, and $x_2^B = 10$. Therefore, the production plans must be $(13.5, 0, -4.5)$ for firm 1 and $(0, 22, -5.5)$ for firm 2.

It doesn't matter what are the shareholdings in this economy. The firms have constant-returns-to-scale technologies, so in any equilibrium, they must make zero profits, and the "dividends" paid to shareholders are zero, no matter how many or how few shares are held.

■ 14.6. Fix an argument $a = (a_1, \ldots, a_n)$. Recall that $a_{\neg\ell}$ is short-hand for $(a_m)_{m \neq \ell}$. For $\ell = 1, \ldots, n$, we know that $A_\ell^*(a_{\neg\ell})$ is nonempty, so some a_ℓ^* lies within $A_\ell^*(a_{\neg\ell})$. But then $(a_1^*, \ldots, a_n^*) \in A^*(a)$; this shows that the A^* correspondence is nonempty valued.

Continue to fix $a = (a_1, \ldots, a_n)$, and suppose $\hat{a}^* = (\hat{a}_1^*, \ldots, \hat{a}_n^*)$ and $\check{a}^* = (\check{a}_1^*, \ldots, \check{a}_n^*)$ are both in $A^*(a)$. This means that, for each ℓ, $\hat{a}_\ell^*$ and $\check{a}_\ell^*$ are in $A_\ell^*(a_{\neg\ell})$. But then for any $\beta \in [0, 1]$, since each A_ℓ^* is convex valued, $\beta\hat{a}_\ell^* + (1 - \beta)\check{a}_\ell^* \in A_\ell^*(a_{\neg\ell})$, and hence $\left(\beta\hat{a}_1^* + (1 - \beta)\check{a}_1^*, \ldots, \beta\hat{a}_n^* + (1 - \beta)\check{a}_n^*\right) \in A^*(a)$. That is, A^* is convex valued.

Suppose $\{a^j\}$ is a sequence (where each a^j has the form $(a_1^j, \ldots, a_n^j)$) with limit $a = (a_1, \ldots, a_n)$, and for each j, $a^{*j} = (a_1^{*j}, \ldots, a_n^{*j}) \in A^*(a^j)$. And suppose $\lim_j a^{*j} = a^*$. For each ℓ from 1 to n, since $a^{*j} \in A^*(a^j)$, we know that $a_\ell^{*j} \in A_\ell^*(a_{\neg\ell}^j)$. And, of course, $\lim_j a_\ell^{*j} = a_\ell^*$. So since each A_ℓ^* is upper semi-continuous, $a_\ell^* \in A_\ell^*(a_{\neg\ell})$. But this implies that $a^* \in A^*(a)$, which demonstrates that A^* is upper semi-continuous.

■ 14.8. Since ζ is a continuous function, it is clear that $\xi : P \to R_+^k$ is continuous. Since $\xi(p) \geq p$, $\sum_{i=1}^k \xi_i(p) \geq 1$, and hence ϕ is continuous. Moreover, $\phi : P \to P$. So by Brouwer's Fixed-Point Theorem (Proposition A8.2), there exists $p \in P$ such that $\phi(p) = p$. I assert that $\phi(p) = p$ implies $\zeta(p) \leq 0$:

Suppose by way of contradiction that $\zeta_i(p) > 0$ for some i. Then $\xi_i(p) > p_i$. Note that this immediately implies that $\sum_\ell \xi_\ell(p) > 1$. If, for this i, $p_i = 0$, then $\phi_i(p) = \xi_i(p)/\sum_\ell \xi_\ell(p) > 0$, contradicting $\phi(p) = p$. So we can assume $p_i > 0$ and hence $p_i\zeta_i(p) > 0$. Since $p \cdot \zeta(p) \leq 0$, there must then be some j such that $p_j\zeta_j(p) < 0$ (to counteract the strictly positive $p_i\zeta_i(p)$ in the dot product), which implies $p_j > 0$ and $\zeta_j(p) < 0$. But then $\xi_j(p) = p_j$, and therefore $\phi_j(p) = \xi_j(p)/\sum_\ell \xi_\ell(p) = p_j/\sum_\ell \xi_\ell(p) < p_j$, which is the final contradiction.

At the risk of stating the obvious: The way this works is to increase the relative prices of goods for which there is positive excess demand. If there are any goods in positive excess demand, this lowers the relative price of goods for which excess demand is negative (as long as the price of the latter good is strictly positive). So at a fixed point of the function, no good can be in positive excess demand.

■ 14.9. The unit simplex of prices P is obviously nonempty, convex, and compact. So that part of the assumptions of Kakutani is clearly satisfied.

We need to show that the correspondence ϕ described in the problem is upper semi-continuous, nonempty-valued, and convex-valued. The construction of ϕ makes it obvious that it both nonempty- and convex-valued; we focus on upper semi-continuity.

Suppose $p^n \rightarrow p$ (in P) and $q^n \in \phi(p^n)$ for each n is such that $\lim_n q^n$ exists and equals q. We must show that $q \in \phi(p)$.

There are two cases to consider. The first case is if p is in the interior of P. Then if $q_i > 0$, $q^n \rightarrow q$ implies that for all sufficiently large n, $q_i^n > 0$. And because $p^n \rightarrow p$ and p is interior to p, it must be that for all sufficiently large n, p^n is interior to P. But if p^n is interior to P, $q^n \in \phi(p^n)$, and $q_i^n > 0$, it must be that $\zeta_i(p^n) \geq \zeta_j(p^n)$ for all j. And since ζ is continuous and $p^n \rightarrow p$, $\zeta_i(p^n) \geq \zeta_j(p^n)$ for all j and all sufficiently large n implies that $\zeta_i(p) \geq \zeta_j(p)$ for all j, which implies that $q \in \phi(p)$.

The second case is where p is on the boundary of P. Let $I \subset \{1, \ldots, k\}$ be the set of indices of p such that $p_i = 0$. We know, of course, that $I \neq \{1, \ldots, k\}$, since for $p \in P$, the sum of its components must be 1. So let $J = \{1, \ldots, k\} \setminus I$; then both I and J are nonempty.

If $q_i > 0$ implies $i \in I$, then $q \in \phi(p)$. So suppose by way of contradiction that $q_i > 0$ but $i \in J$, meaning $p_i > 0$. Because $q^n \rightarrow q$ and $p^n \rightarrow p$, we know that for all sufficiently large n, $q_i^n > 0$ and $p_i^n > 0$. More than that, we know that for large n, p_i^n is strictly bounded away from zero; that is, there exists some $\epsilon > 0$ such that $p_i^n \geq \epsilon$ for all large n. Now it cannot be true that (for large n) p^n is on the boundary of P, for if it were, $p_i^n > 0$ and $q^n \in \phi(p^n)$ would imply that $q_i^n = 0$. So we know that for all sufficiently large n, $q_i^n > 0$, $p_i^n \geq \epsilon$, and p^n lies in the interior of P.

But $p^n \cdot \zeta(p^n) = 0$, and there is a uniform lower bound on all components of $\zeta(p)$ for all p. If $-B$ is that lower bound (and wlog, we can assume $B > 0$), then $p_i^n \zeta_i(p^n) \leq B(k-1)$, since the other $k-1$ terms in the dot product $p^n \cdot \zeta(p^n)$ are bounded below by $1 \times (-B)$. Hence $\zeta_i(p^n) \leq B(k-1)/\epsilon$ for all sufficiently large n.

However, since $p^n \rightarrow p$ and some components of p are zero, we know that $\lim_n \sum_{j \in I} \zeta_j(p^n) = \infty$. This means that for all sufficiently large n, one of the components of $\zeta_j(p^n)$ for $j \in I$ must exceed $B(k-1)/\epsilon \geq \zeta_i(p^n)$, hence q_i^n must equal zero, a contradiction. Therefore, ϕ is indeed upper semi-continuous.

All the conditions of Kakutani's Fixed-Point Theorem hold, and we know there is a fixed point: For some $p^* \in P$, $p^* \in \phi(p^*)$. Obviously, this p^* cannot be on the boundary of P. For if $p_i^* > 0$ while $p_j^* = 0$ for some $j \neq i$, then the rules by which ϕ has been constructed tell us that $q_i = 0$ for all $q \in \phi(p^*)$. And if p^* is interior to P, then $p_i^* > 0$ for all i. But then by the rule by which ϕ has been constructed, $\zeta_i(p^*) \geq \zeta_j(p^*)$ for all i and j. That is, $\zeta_i(p^*) = \zeta_j(p^*)$ for all i and j. And since $p^* \cdot \zeta(p^*) = 0$ and all the components of p^* are strictly positive, this entails $\zeta_i(p^*) = 0$ for all components, or $\zeta(p^*) = 0$; p^* is an equilibrium price vector.

■ 14.11. If you possess an early printing of the book, the version of Proposition 14.14 in your book asserts that $e \Rightarrow \mathbf{W}(\mathcal{E}(e))$ is upper semi-continuous, as long as eqquilibrium prices are known to be nonnegative and nonzero. Later printings of the book have been corrected: This is not true, and we only (necessarily) get upper semi-continuity at limit points where prices are strictly positive. (Also, if you have an earlier printing, the statement of Problem 14.11 lacks an asterisk, so you might be surprised to find its solution here. Since I screwed up on the statement of the proposition in earlier printings, I decided it would be a good idea to move the discussion to the *Student's Guide*.)

Let me first prove the version of the proposition that is given in later printings. For those with an earlier printing, this is the assertion that, if $\{e_n\}$ is a sequence of endowments with limit e, if $(p_n, \boldsymbol{x}_n, \boldsymbol{z}_n)$ is a Walrasian equilibrium for the economy $\mathcal{E}(e_n)$, and if $\lim_n (p_n, \boldsymbol{x}_n, \boldsymbol{z}_n) = (p, \boldsymbol{x}, \boldsymbol{z})$ where p is strictly positive, then $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for the economiy $\mathcal{E}(e)$.

There are four things to show. First is that markets clear in the limit economy. But market clearing in the nth economy is

$$\sum_h \boldsymbol{x}_n^h \leq \sum_h e_n^h + \sum_f \boldsymbol{z}_n^h,$$

and passing to the limit in n gives us market clearing for the limit economy. Second, for each firm f, $\boldsymbol{z}^f$ must maximize profit for f at prices p. Let $\hat{z}$ be any (other) production plan from Z^f. The profit-maximization condition for firm f in the nth economy implies that $p_n \cdot \boldsymbol{z}_n^f \geq p_n \cdot \hat{z}$, and passing to the limit in n gives $p \cdot \boldsymbol{z}^f \geq p \cdot \hat{z}$.

The third condition is that consumer h can afford $\boldsymbol{x}^h$ at prices p and given the production plans specified by $\boldsymbol{z}^f$. But this condition for the nth economy is that

$$p_n \cdot \boldsymbol{x}_n^h \leq p_n \cdot e_n^h + \sum_f s^{fh} p_n \cdot \boldsymbol{z}_n^f,$$

and passing to the limit in n once more gives us what we want.

And, finally, we must show that for each h, $\boldsymbol{x}^h$ is preference/utility maximizing for h subject to h's budget constraint in the limit economy. I'll do this in two steps.

Step 1. *If* $\hat{x} \succ^h \boldsymbol{x}^h$, *then* $p \cdot \hat{x} \geq p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$. Or, in words, any bundle that is strictly preferred to $\boldsymbol{x}^h$ costs at least as much at the limit prices p as h has to spend. To see this, note that if $\hat{x} \succ^h \boldsymbol{x}^h$, then (since $\boldsymbol{x}^h = \lim_n \boldsymbol{x}_n^h$) there exists N such that, for all $n \geq N$, $\hat{x}^h \succ^h \boldsymbol{x}_n^h$. Hence for all large n,

$$p_n \cdot \hat{x} > p_n \cdot e_n^h + \sum_f s^{fh} p_n \cdot \boldsymbol{z}_n^f. \qquad \text{(SG14.1)}$$

And (one more time!) passing to the limit in n gives the desired result. (For future reference, note that while (SG14.1) has a strict inequality, passing to the limit makes the limit inequality weak. However, if instead of the strict inequality in (SG14.1) we had only weak inequalities, we would still get a weak inequality in the limit.)

Step 2. *If* $\hat{x} \succ^h \boldsymbol{x}^h$*, then* $p \cdot \hat{x} > p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$. That is, in the first step we showed that any bundle better than $\boldsymbol{x}^h$ would cost h at least as much as her full resources; now we will show that it costs strictly more. Suppose it does not cost more; then in view of the first step, we must be dealing with a case where $p \cdot \hat{x} = p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$.

Since $\boldsymbol{x}^h \in R^k_+$ and p is nonnegative, $p \cdot \boldsymbol{x}^h \geq 0$. This implies that $p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f \geq 0$; that is, h has (in the limit) nonnegative wealth. Suppose that $p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f = 0$. Then in the limit economy, h has zero wealth, and the only feasible (nonnegative) bundle she can afford is the zero bundle. If $\hat{x} \succ^h \boldsymbol{x}^h = 0$, then $\hat{x}$ cannot be the zero bundle. But it must be nonnegative, so since prices p are strictly positive, $p \cdot \hat{x} > 0$, contradicting the hypothesis that $\hat{x}$ costs precisely h's financial resources.

Or it may be that $0 < p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$ (which, $= p \cdot \hat{x}$). Consider bundles $\alpha \hat{x}^f$ for α less than but close to 1. By continuity of h's preferences, for some such α, $\alpha \hat{x} \succ^h \boldsymbol{x}^h$. Since $p \cdot \hat{x} = p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f > 0$, for $\alpha < 1$, $p \cdot (\alpha \hat{x}) < p \cdot \hat{x} = p \cdot e^h + \sum_f s^{fh} p \cdot \boldsymbol{z}^f$, contradicting what we showed in Step 1.

So we know that if $\hat{x} \succ^h \boldsymbol{x}^h$, $\hat{x}$ must cost more at prices p than h can afford, and $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for the economy $\mathcal{E}(e)$. ∎

That finishes the proof of the Proposition (as stated, correctly, in later printings of the book.) But it is worth saying a few things more. Suppose we are in a setting where equilibrium prices are always nonnegative and nonzero. In this case, we can normalize any price vector p to lie in the unit simplex; for the remainder of this discussion, do so.

In the next chapter, the concept of a *Walrasian quasi-equilibriun* is defined. You can read the formal definition in the text in Definition 15.3; the short description is that this is a triple of (nonnegative, non-zero) prices p, consumption allocations $\boldsymbol{x}$, and production plans $\boldsymbol{z}$, such that (a) markets clear, (b) firms maximize their profit, (c) consumers can afford their part of the consumption allocation, and (d) any bundle $\hat{x}^h$ that is preferred by agent h to her allocated $\boldsymbol{x}^h$ costs (at prices p) at least as much as h's resources (given p, her endowment, her shareholdings, and the firms' production plans). This is the same as a Walrasian equilibrium, except that part d is a bit weaker; in a Walrasian equilibrium, if $\hat{x} \succ^h \boldsymbol{x}^h$, then $\hat{x}$ must cost strictly more than h's financial resources. Of course, this implies that every Walrasian equilibrium is also a Walrasian quasi-equilibrium.

Suppose that we let $\boldsymbol{Q}(\mathcal{E}(e))$ be the set of Walrasian quasi-equilibria for the economy $\mathcal{E}(e)$, where we look only at nonnegative, nonzero prices, that are normalized to be in the unit simplex of prices. Then I assert that the proof of Proposition 14.14 just given is easily adapted to show that $e \Rightarrow \boldsymbol{Q}(\mathcal{E}(e))$ is upper semi-continuous. Go through the steps, if this isn't clear to you; the key step is connected to the parenthetical remark made at the end of Step 1.

So why isn't $e \Rightarrow \mathbf{W}(\mathcal{E}(e))$ upper semi-continuous? What is wrong with the proposition as given in early printings of the book? There are two problems, one obvious and easily repaired, but the other requiring that we move from $\mathbf{W}$ to $\boldsymbol{Q}$.

The first, easily fixed problem concerns the normalization of prices. If $(p, \boldsymbol{x}, \boldsymbol{z})$ is a Walrasian equilibrium for $\mathcal{E}(e)$, then (of course) so is $(\lambda p, \boldsymbol{x}, \boldsymbol{z})$ for any strictly positive scalar λ. So if we don't normalize prices, we could have a sequence of equilibria $\{(p_n, \boldsymbol{x}_n, \boldsymbol{z}_n)\}$ where the price vectors converge to 0. Of course, those limit prices will not be Walrasian equilibrium prices (at least, as long as consumers are, say, globally insatiable). But the fix is easy: Be in a setting where the only prices considered are nonnegative and nonzero, and look (only) at prices normalized to be in the unit simplex. Then any limit price vector will be nonnegative, nonzero, and (in fact) in the unit simplex.

But the second, harder-to-fix problem, involves consumers whose wealth goes to zero. Let me provide a concrete example of a two-consumer, two-good, pure-exchange economy: The two consumers are Alice and Bob. Alice will have endowment $e^A = 0$ throughout. We can give Alice absolutely standard preferences: $u^A((x_1, x_2)) = x_1^{1/2} + x_2^{1/2}$, say. Bob's preferences will be quasi-linear in good 2, $u^B(x_1, x_2) = x_1^{1/2} + x_2$. And Bob's endowment in economy n will be

$$e_n^B = \left(\frac{1}{n}, 1 - \frac{1}{n^{1/2}}\right).$$

Note that this is set up so that Bob's endowment gives him utility 1 for all n; we are riding along his $u^B = 1$ indifference curve, decreasing to zero the amount of good 1 and compensating with more and more good 2.

I assert that, with this endowment, equilibrium prices in economy n (normalized to be in the unit simplex) are

$$p_n = \left(\frac{n^{1/2}}{2 + n^{1/2}}, \frac{2}{2 + n^{1/2}}\right).$$

At these prices, Alice (whose wealth is zero, because her endowment is 0), optimally chooses $\boldsymbol{x}_n^A = 0$, while Bob chooses his endowment $x_n^B = e_n^B$. There is nothing mysterious in this construction; since Alice has zero endowment, as long as prices are strictly positive, she will get 0, so Bob must be picking his endowment. And the equilibrium prices are chosen so that they support Bob's choice of his endowment.[1]

But note that in p_n, the price of the first good is getting close to 1, while the price of the second good is getting close to 0. So

$$\lim_{n\to\infty}(p_n, \boldsymbol{x}_n^A, \boldsymbol{x}_n^B) = (p, \boldsymbol{x}^A, \boldsymbol{x}^B) \quad \text{where} \quad p = (1,0),\ \boldsymbol{x}^A = (0,0),\ \text{and } \boldsymbol{x}^B = (0,1).$$

This clearly is not a Walrasian equilibrium for the limit economy (in which Bob's endowment is $e^B = (0,1)$); markets clear, but neither Alice nor Bob are satisfied with their assigned consumption bundles; since good 2 is free, both of them want infinite amounts of it.

But, of course, $(p, \boldsymbol{x}^A, \boldsymbol{x}^B)$ is a Walrasian quasi-equilibrium. The bundles that both Alice and Bob prefer and can afford cost the same as their equilibrium allocations, since these strictly preferred bundles involve large quantities of a good whose price is zero.

This is not to say that a triple $(p, \boldsymbol{x}, \boldsymbol{z})$ cannot be a (full-fledged) Walrasian equilibrium, if p has some zero components. But if we want to save the result that $e \Rightarrow \mathbf{W}(\mathcal{E}(e))$ is upper semi-continuous, we either must find ways to ensure that equilibrium prices are bounded uniformly away from zero (on the unit simplex), or we must make sure that the price of a good can go to zero only in situations where eveyone is satiated in the good.[2]

[1] Since both Alice and Bob have strictly increasing preferences, prices have to be strictly positive in equilibrium. As long as Alice has endowment 0, at strictly positive prices, she must choose 0, so Bob must consume his endowment, and so equilibrium prices must support this choice for him. That is to say, we have described the *only* Walrasian equilibrium for economy $\mathcal{E}(e_n)$. It may be worth noting a couple of things here: (1) If $e^A = (0,0)$ and $e^B = (0,1)$, there is no Walrasian equilibrium at all. There are a number of existence results in this chapter, and you might want to go through them all and figure out why none of them work in this case. (2) This could have been posed as a one-consumer, pure-exchange economy—Alice plays no real role in the counterexample—but I worried that readers might think that a one-person economy was somehow a "trick," so poor Alice was included.

[2] As an example of the first approach, Phil Reny suggests that we constrain endowments and production technologies so that there is a uniform upper bound on what the economy can allocate, and then assume that consumer preferences are such that the norm of aggregate consumption demand goes to infinity for any sequence of prices that approaches the boundary of P. Then Walrasian equilibrium prices must stay uniformly away from the boundary, for any sequence of endowments $\{e_n\}$. If, by way of contradiction, we had a sequence of endowment vectors $\{e_n\}$ and Walrasian equilibrium prices p_n for those endowment vectors, net consumer demand would have to diverge in norm. This would mean that, for large n, net consumer demand (in equilibrium) would have to exceed what the economy is (uniformly) capable of producing. That, of course, cannot happen.