Microeconomic Foundations I
Choice and Competitive Markets

Student's Guide

Chapter 15: General Equilibrium, Efficiency and the Core

Summary of the Chapter

Since Adam Smith's *Wealth of Nations*, the idea that market outcomes are socially "good" (at least, compared to other ways of organizing the production and distribution of goods) has been advanced by economists. There is a lot to this story, concerning innovation and growth, for which we lack important tools of analysis. But, in this chapter, we begin to understand this notion, with formal results that show:

- Walrasian-equilibrium allocations are Pareto efficient, the *First Theorem of Welfare Economics*.
- Conversely (and subject to some extra convexity assumptions and other qualifications), every Pareto-efficient allocation can be "decentralized" as a Walrasian-equilibrium allocation, if endowments and shareholdings redistributed, the *Second Theorem of Welfare Economics*.
- Moreover, every Walrasian-equilibrium allocation is in an appropriately defined core.
- With "enough" competition of the "right kind," core allocations are Walrasian-equilibrium allocations. This idea is formalized in the literature in several ways; we provide one formalization, the *Debreu-Scarf Theorem*.

It goes without saying that these results are not tautologies, and for them to hold, certain conditions must hold. For one thing, all economic entities—consumers and firms—must be price-takers. For another, there can be no externalities; the chapter closes with a brief discussion of externalities and *Lindahl equilibria*, an extension of Walrasian equilibria that is designed to get back the Pareto efficiency of equilibrium allocations in economies with externalities, by adding many more markets to the economy.

Solutions to Starred Problems

■ *2. (a) At the prices $(1, 1)$ the symmetry and strict concavity of Alice's utility function implies that she will choose $x_1 = x_2$. Her preferences are strictly increasing, hence locally insatiable, and so she will satisfy her budget constraint. Hence her unique demand at these prices is $(1, 1)$.

As for Bob: The function $x^3 - 9x^2 + 15x$ is increasing at $x = 0$, hits a local maximum at $x = 1$ where its value is 7, then declines until $x = 5$, and then increases to ∞. This means that, for Bob, $(1, 1)$ is a local maximum. Moreover, for the price vector $(1, 1)$, Bob's wealth is 4, so he cannot afford any bundle that does better than $(1, 1)$. That is his (unique) demand.

So total demand is $(2, 2)$, while the social endowment is $(3, 3)$, and markets clear.

(b) But this allocation is not Pareto efficient. Since Alice has strictly increasing utility, it is optimal to give her anything Bob doesn't want. The equilibrium allocation is Pareto inferior to the allocation that leaves Bob at $(1, 1)$ and gives Alice the remaining $(2, 2)$.

(c) A key step in the proof of the First Theorem is to say that all Pareto-superior allocations correspond to bundles of goods that cost more than the consumer's wealth. While it is true, in this case, that allocations Pareto superior to the equilibrium allocation cost more than the equilibrium allocation, because Bob is not locally insatiable, his equilibrium allocation costs less than all his wealth. The proof breaks down.

(Can you produce an example where both consumers spend all their wealth at the equilibrium and yet the equilibrium allocation is not Pareto efficient? As a starting point, what is the value of x for $x > 5$ for which $x^3 - 9x^2 + 15x = 7$. Why is that significant?)

■ 4. See Figure G15.1. In this figure, consumer 1's indifference curves as $x_1 \to 0$ hit the axis, but do so at a slope that approaches infinity. Consumer 2, on the other hand, has no use for good 2; consumer 2 wants (only) as much good 1 as he can get. (In the figure, consumer 2's indifference curves are shown as dashed lines.) Hence Pareto-efficient points give all of good 2 to consumer 1. The social endowment of good 1 can be divided between them, but one Pareto-efficient point, and the one that is going to be problematic, is where consumer 1 gets all of good 2 while consumer 2 gets all of good 1. The prices that support this allocation are (multiples of) $p = (p_1, p_2) = (1, 0)$. But at these prices, consumer 1 wants an unlimited amount of good 2; bundles that give consumer 1 more than the social endowment of good 2 are strictly preferred to her (quasi-)equilibrium allocation, but cost just as much (namely, \$0).

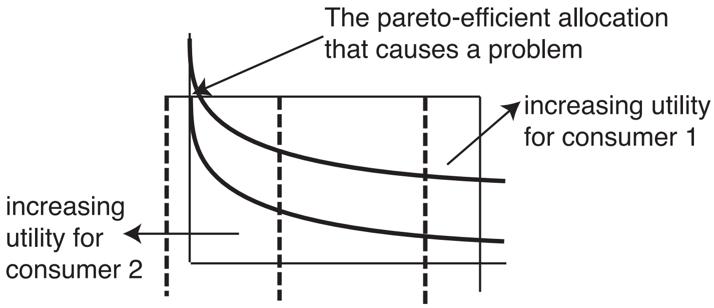

Figure G15.1. A quasi-equilibrium that is not a Walrasian equilibrium.

■ 5. If $\boldsymbol{x}$ is not in the core, then there must be some coalition J and some allocation $(\hat{x}_j)_{j \in J} \in \boldsymbol{X}^J$ such that $\hat{x}^j \succeq^j \boldsymbol{x}^j$ for all $j \in J$ and $\hat{x}^j \succ^j \boldsymbol{x}^j$ for some $j \in J$. But if $\hat{x}^j \succeq^j \boldsymbol{x}^j$, then $p \cdot \hat{x}^j \geq p \cdot e^j$; otherwise, consumer j could afford something close to and better than $\hat{x}^j$ at prices p, but then that bundle would be better than $\boldsymbol{x}^j$, which is not possible since $\boldsymbol{x}^j$ solves consumer j's utility-maximization problem at prices p. And for any j such that $\hat{x}^j \succ^j \boldsymbol{x}^j$, $p \cdot \hat{x}^j > p \cdot e^j$. Summing up over all the j, we have

$$p \cdot \left[\sum_{j \in J} \hat{x}^j\right] > p \cdot \left[\sum_{j \in J} e^j\right].$$

But if $(\hat{x}^j)_{j \in J} \in \boldsymbol{X}^J$, then $\sum_{j \in J} \hat{x}^j \leq \sum_{j \in J} e^j$. Prices p are nonnegative, so this implies that

$$p \cdot \left[\sum_{j \in J} \hat{x}^j\right] \leq p \cdot \left[\sum_{j \in J} e^j\right],$$

a direct contradiction to what we had before.

■ 7. The proof of Proposition 15.9 mimics that of Proposition 15.8 (see the solution to Problem 15.5 just above), except that you must take into account the profits of firms. Suppose first that $\boldsymbol{X}^J$ is defined using rule 5 for each J. We begin as in the solution to Problem 15.5:

If $\boldsymbol{x}$ is not in the core, then there must be some coalition J and some allocation $(\hat{x}_j)_{j \in J} \in \boldsymbol{X}^J$ such that $\hat{x}^j \succeq^j \boldsymbol{x}^j$ for all $j \in J$ and $\hat{x}^j \succ^j \boldsymbol{x}^j$ for some $j \in J$. But if $\hat{x}^j \succeq^j \boldsymbol{x}^j$, then $p \cdot \hat{x}^j \geq p \cdot e^j + \sum_f s^{fj} p \cdot \boldsymbol{z}^f$; otherwise, consumer j could afford something close to and better than $\hat{x}^j$ at prices p, but then that bundle would be better than $\boldsymbol{x}^j$, which is not possible since $\boldsymbol{x}^j$ solves consumer j's utility-maximization problem at prices p. And for any j such that $\hat{x}^j \succ^j \boldsymbol{x}^j$, $p \cdot \hat{x}^j > p \cdot e^j + \sum_f s^{fj} p \cdot \boldsymbol{z}^f$. Summing up over all the j, we have

$$p \cdot \left[\sum_{j \in J} \hat{x}^j\right] > p \cdot \left[\sum_{j \in J} e^j\right] + p \cdot \left[\sum_{j \in J} \sum_f s^{fj} \boldsymbol{z}^f\right] = p \cdot \left[\sum_{j \in J} e^j\right] + p \cdot \left[\sum_f s^{fJ} \boldsymbol{z}^f\right].$$

But if $(\hat{x}^j)_{j\in J} \in \boldsymbol{X}^J$, then $\sum_{j\in J}\hat{x}^j \le \sum_{j\in J} e^j + \sum_f s^{fJ}\hat{z}^f$, for some production plans $(\hat{z}^f)_{f\in F}$, since $\boldsymbol{X}^J$ is defined using rule 5. Prices p are nonnegative, so this implies that

$$p\cdot\left[\sum_{j\in J}\hat{x}^j\right] \le p\cdot\left[\sum_{j\in J}e^j\right] + p\cdot\left[\sum_f s^{fJ}\hat{z}^f\right].$$

And since $\boldsymbol{z}^f$ is profit maximizing for firm f at prices p, and $\hat{z}^f$ is a feasible production plan for firm f, we maintain the inequality just above if we replace each $\hat{z}^f$ with $\boldsymbol{z}^f$, giving us a direct contradiction to the inequality obtained two displays ago.

Since $\boldsymbol{X}^J$ under rules 1 and 3 is smaller than $\boldsymbol{X}^J$ under rule 5, this proof shows that every Walrasian equilibrium is in the core of the economy, if rules 1 or 3 are used. (The assumption that $0 \in Z^f$ for each f needs to be used here. Can you see why? If this is not assumed, why might rule 1 make it *easier* for a coalition to construct a valid objection to an equilibrium allocation?)

And if any of the rules is used, and all firms have constant-returns-to-scale technologies: Note first that, since $0 \in Z^f$ and all firms have c-r-s technologies, $p\cdot\boldsymbol{z}^f = 0$ for all firms f in any Walrasian equilibrium. (In fact, if I assume that all technologies are c-r-s, I don't need the assumption that $0 \in Z^f$ for each f. Why?) Rules 1, 3, and 5 are covered by the previous argument, so we only need to worry about rules 2 and 4. Rule 2 is the more liberal (gives more power to each coalition to form objections), so we can work with that specification of $\boldsymbol{X}^J$. Then at any Walrasian-equilibrium prices, $p\cdot\boldsymbol{z}^f = 0$ for the profit-maximizing plans $\boldsymbol{z}$, and $p\cdot\hat{z}^f \le 0$ for any other plans. Therefore, even if the most liberal rule 2 is used, feasibility of objection $(\hat{x}^j)$ implies that

$$\sum_{j\in J}\hat{x}^j \le \sum_{j\in J}e^j + \sum_{f\in F}\hat{z}^f,$$

for some production plan $(\hat{z}^f)_{f\in F}$. Evaluating both sides of the inequality with equilibrium prices p, and noting that $p\cdot z^f \le 0$ for any feasible production plan for firm f, gives

$$p\cdot\left[\sum_{j\in J}\hat{x}^j\right] \le p\cdot\left[\sum_{j\in J}e^j\right] + p\cdot\left[\sum_{f\in F}\hat{z}^f\right] \le p\cdot\left[\sum_{j\in J}e^j\right].$$

And, in the first part of argument, the consumer-by-consumer inequalities $p\cdot\hat{x}^j \ge p\cdot e^j + \sum_f s^{fj}p\cdot\boldsymbol{z}^f$ become $p\cdot\hat{x}^j \ge p\cdot e^j$, since the sum of the firm's profits is just a sum of zeros. Adding up over all consumers (and noting that some of the inequalities are strict) gives the reverse inequality, and the (by now) usual contradiction.

■ 11. (a) This is a Chapter-8 style problem. Both utility functions are concave functions of the social state (an allocation of the social endowment), and the set of social states is convex, so we find all the Pareto-efficient points by maximizing weighted averages of the two utility functions (making due allowance for the "endpoints" of such weighted averages).

If we give all the weight to Alice, the obvious Pareto-efficient allocation provides Alice with all the goods, or $(x^A, x^B) = ((4,4,),(0,0))$.

If we give all the weight to Bob, it is not so simple. It is clear that Bob should get all of good 2. But Bob gets utility from Alice consuming good 1, so we have to solve $\max \ln(x_1^B + 1) + 0.5\ln(x_1^A + 1)$ subject to nonnegativity constraints and $x_1^A + x_1^B \leq 4$. Simple calculus reveals an answer of $x_1^B = 3$; the Pareto-efficient allocation corresponding to this weighting is $(x^A, x^B) = ((1,0),(3,4))$.

And if there is positive weight given to each: Normalize the weight on Bob to be 1, and let $\alpha \in (0,\infty)$ be the relative weight on Alice. The problem is to

$$\text{maximize}\, \alpha\big(\ln(x_1^A + 1) + \ln(x_2^A + 1)\big) + \big(\ln(x_1^B + 1) + \ln(x_2^B + 1) + 0.5\ln(x_1^A + 1)\big),$$

subject to nonnegativity constraints and $x_1^A + x_1^B \leq 4$ and $x_2^A + x_2^B \leq 4$. Since the utility functions are strictly increasing in the variables, we know that the solution will have $x_1^B = 4 - x_1^A$ and $x_2^B = 4 - x_2^A$, so you can replace x_1^B and x_2^B in the objective function with $4 - x_1^A$ and $4 - x_2^A$, respectively. Also, the objective function is (strictly) concave in the variables, so we know that the first-order/complementary slackness conditions are necessary and sufficient for an optimum. Finally, we know that if the unconstrained maximum violates one of the constraints, the answer will be at the violated constraint. (This follows from the concavity of the objective function.) So we look at the simple first-order conditions on the objective function. We get

$$x_1^A = \frac{10\alpha + 3}{2\alpha + 3} \quad \text{and} \quad x_A^2 = \frac{5\alpha - 1}{\alpha + 1}.$$

Note that for some values of α, these equations violate the constraints, so the "real" solution must be amended. Specifically, for large α (lots of weight on Alice), we get the simple first-order conditions of x_1^A and x_2^A both approaching 5. When these reach 4, there is no more to give Alice. And as α approaches zero (most of the weight on Bob), there is no problem with x_1^A, which approaches 1, but x_2^A approaches -1, and this has to stop at 0. Putting all this together, we get for the full Pareto frontier:

$$x^A, x^B = \begin{cases} \left(1, 0\right), \left(3, 4\right), & \text{for } \alpha = 0, \\ \left(\dfrac{10\alpha + 3}{2\alpha + 3}, 0\right), \left(\dfrac{9 - 2\alpha}{2\alpha + 3}, 4\right), & \text{for } 0 < \alpha \leq 1/5, \\ \left(\dfrac{10\alpha + 3}{2\alpha + 3}, \dfrac{5\alpha - 1}{\alpha + 1}\right), \left(\dfrac{9 - 2\alpha}{2\alpha + 3}, \dfrac{5 - \alpha}{\alpha + 1}\right), & \text{for } 1/5 \leq \alpha \leq 9/2, \\ \left(4, \dfrac{5\alpha - 1}{\alpha + 1}\right), \left(0, \dfrac{5 - \alpha}{\alpha + 1}\right), & \text{for } 9/2 \leq \alpha \leq 5, \text{ and} \\ \left(4, 4\right), \left(0, 0\right), & \text{for } \alpha \geq 5, \text{ including } \alpha = \infty. \end{cases}$$

(b) To find the Walrasian equilibria, we proceed as usual, simply ignoring the externality in Bob's utility function. Bob is going to take Alice's choices as something he cannot affect, and so that term enters his utility-maximization problem as a constant.

Given that this is so, all the symmetry and strict convexity (of preferences) ensures that the equilibrium will be symmetric, meaning that both Alice and Bob wind up with equilibrium allocations of $(2, 2)$. To support these allocations, normalized prices must be $(1, 1)$.

This allocation is (of course) *not* Pareto efficient. To allocate 2 units of good 1 to Alice along the Pareto frontier, we are looking at α in the range $\alpha \in (0, 9/2)$, for which $x_1^A = (10\alpha + 3)/(2\alpha + 3)$. Setting this equal to 2 gives $\alpha = 1/2$. Note well, $\alpha = 1/2$ is the only point along the Pareto frontier for which $x_1^A = 2$. But when $\alpha = 1/2$, the corresponding point on the Pareto frontier is $(x^A, x^B) = ((2, 1), (2, 3))$.

Now, this point is *not* a Pareto improvement on the equilibrium allocation, because it decreases Alice's utility. We've only demonstrated that the Walrasian equilibrium allocation is not on the Pareto-frontier that we computed in part a. A different way to see that this point is not Pareto efficient is to move from it (from $((2, 2), (2, 2))$) in Pareto-improving direction, and this means having Bob give up some of his 1-good to Alice while getting some 2-good in return. Suppose Bob gives up 0.1 units of the first good, and gets back 0.08 units of the second good in return. At the equilibrium allocation, Alice's utility is 2.19722458 and Bob's is 2.74653072, while at this alternative, Alice has utility 2.20298573 and Bob has 2.75534139. So this is a Pareto improvement. It isn't hard to see why. The terms of trade—.0.1 units of good 2 in exchange for 0.08 units of good 1—are good for Alice. And while they diminish those portions of Bob's utility which bear directly on his own compensation, the external effect of having Alice consume more of good 1 more than makes up for this.

(c) There is only one external effect here, so we need to take into account only one "extra" price in the Lindahl equilibrium, a transfer from Alice to Bob because of the external impact on Bob of Alice's consumption of good 1. Let this have a price r, and let the "regular prices" of goods 1 and 2 be normalized to be 1 and p, respectively. Alice's Lindahl problem is then

$$\max \ln(x_1^A + 1) + \ln(x_2^A + 1), \text{ subject to } (1 + r)x_1^A + px_2^A \leq 4.$$

Alice maximizes her utility, but in terms of her budget constraint, it "costs" her r per unit of the first good that she consumes.

As for Bob, his problem is

$$\max \ln(x_1^B + 1) + \ln(x_2^B + 1) + 0.5 \ln(x_1^A + 1), \text{ subject to } x_1^B + px_2^B \leq 4p + rx_1^A.$$

He also maximizes his utility, but for him, a change in x_1^A is a transfer to him, so we put this on the "income" side of his budget constraint.

Now look at the following prices: $p_1 = 1$, $p_2 = 0.82057582$, and $r = -0.27984671$. Note that $r < 0$; the activity x_1^A generates a positive externality for Bob, and so the transfer from Alice to Bob for this activity should be negative; that is, Bob should compensate (or, a better verb is, subsidize) Alice for her consumption of the first good.

If you run the math, you'll find that, at these prices, Alice chooses (2.84690988, 2.37612289) and Bob chooses $(1.15309012, 1.62387711)$, which clears markets. And, just as Proposition 15.13 tells us, this allocation is Pareto efficient, corresponding to $\alpha = 1.28669246$.

How did I find this equilibrium? One method you might try is to write down the optimal solutions to Alice's and Bob's problems as a function of the price vector $(1, p, r)$, and then see what it takes to get markets to clear. I tried that, and I have to admit that the algebra defeated me. It is certainly possible to do, but I kept making errors that led me astray.[1] But then it occurred to me to use Proposition 15.13. I know that the equilibrium allocation has to be Pareto efficient, so it must correspond to some value of α. I also know that, with only one positive externality in the problem, the ratios of the supporting prices have to match Bob's marginal utilities for the *three* variables in his utility function. That is, the full price vector (p_1, p_2, r) must be proportional to

$$\left(\frac{1}{x_1^B + 1}, \frac{1}{x_2^B + 1}, -\frac{0.5}{x_1^A + 1} \right).$$

(I knew that r had to be negative, because Bob wants to subsidize Alice's consumption of good 1.) So for a variety of values of α, I computed the allocations and then those relative prices, and then for each allocation and set of relative prices, I computed Bob's

[1] But see the next paragraph.

budget: the value of his endowment "less" the amount he subsidizes Alice, less the cost of his purchases. At the value of α (namely, 1.28...) where his budget just balanced, I knew I had my equilibrium. (Of course, I then checked that if Alice and Bob maximize at those relative prices, it all works. I did this numerically, and it all worked perfectly.)

Added after the fact: While I think that the method for finding the equilibrium in the previous paragraph is interesting enough to be retained, Alejandro Francetich (who, as a Ph.D. student, did a magnificent job of going carefully through the various pieces of the text and ancilliaries, finding numerous typos and think-os) was able to do the algebra needed for a direct solution of the equilibrium. Here is his solution:

■ 15.12. The front-cover design gives a graphical depiction of the proof of the Debreu-Scarf Theorem for the case of two commodities and two consumers. On the left-hand side, an Edgeworth box is given, with endowment point e and a reallocation $\boldsymbol{x}$ of the endowment, together with indifference curves for the two consumers through $\boldsymbol{x}$. Since this is an Edgeworth box depiction, we know that the reallocation is non-wasteful; if we call the consumers Alice and Bob and we denote their endowments e^A and e^B and their portions of $\boldsymbol{x}$ by $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$, then we know that $\boldsymbol{x}^A + \boldsymbol{x}^B = e^A + e^B$.

On the right-hand side of the cover, we depict the sets of net trades (from their endowments) for Alice and Bob that, together with those endowments, provide them with more utility than do $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$, respectively. Note that $\boldsymbol{x}^A - e^A$, the red dot in the north-west quantrant, is on the boundary of this set for Alice, while $\boldsymbol{x}^B - e^B = e^A - \boldsymbol{x}^A$ (since the reallocation $(\boldsymbol{x}^A, \boldsymbol{x}^B)$ is non-wasteful) lies on the boundary of this set for Bob. The full boundary for Alice just takes her (blue) indifference curve from the Edgeworth box and translates it onto the axes on the right-hand side; for Bob, we take his (yellow) indifference curve from the Edgeworth box, rotate is 180 degrees, and translate it appropropriately. And then we form the convex hull—the shaded green area, bounded by the dashed green line—of these two regions.

Since this convex hull is disjoint from the strict negative orthant (the dark green area containing the author's name), we know (from the proof) that the reallocation $\boldsymbol{x}$ is a Walrasian-equilibirum allocation relative to e, with corresponding relative prices given by the dashed green line, which is the hyperplane that separates the convex hull from the strict negative orthant.

Note that, since $\boldsymbol{x}^A + \boldsymbol{x}^B = e^A + e^B$, or $\boldsymbol{x}^A - e^A = -(x^B - e^B)$, and since (in the depiction) preferences are clearly strictly increasing, we know that $\boldsymbol{x}^A - e^A$ is on the boundary of Alice's set of strictly preferred net trades, while $\boldsymbol{x}^B - e^B$ is on the boundary of Bob's set. Hence $(1/2)(\boldsymbol{x}^A - e^A + x^B - e^B) = (1/2)0 = 0$ is (at least) on the boundary of the convex hull. And the only way the origin will be on the boundary—that is, not in the interior, which would mean that there is intersection with the strict negative orthant—is if the line formed by joining $\boldsymbol{x}^A - e^A$ to $\boldsymbol{x}^B - e^B$ is tangent to the two sets of strictly preferred net trades. But this would mean (a) the indifference curves back in the Edgeworth box must be tangent to one another, and (b) the line of tangency must pass through the endowment point. Which is, of course, the condition for $\boldsymbol{x}$ to be a Walrasian-equilibrium allocation relative to e, framed in terms of the Edgeworth box picture.